2nd Order Peano Arithmetic

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Motivation for Second-Order Logic

Let P be a monadic predicate. Then the 1st-order sentence

$$\exists x(Px \to \forall x Px)$$

is a logical truth. One is then tempted to generalize by saying, not just in the metalanguage that this is so for any choice of P, but in the object language as well:

$$\forall P \exists x(Px \to \forall x Px).$$

This is to introduce predicate variables and quantification over them. One might then replace the induction schema in Peano Arithmetic by a single wff (call it the *Induction Axiom*):

$$\forall P((P\mathbf{0} \land \forall x(Px \to P\mathbf{S}x)) \to \forall x Px).$$

Might moving to 2nd order logic help with incompleteness?

2nd Order Syntax

For each $n > 0$, let our n-ary predicate variables be given by

$$X_0^n, X_1^n, X_2^n, \dots, X_i^n, \dots.$$

The notion of a 2nd-order language is to take the non-logical symbols of a first order language $\mathcal{L}$, and to revise the definition of wff and sentence by using 2nd-order predicate variables and quantification over them. (So what changes is not $\mathcal{L}$, but $Wff(\mathcal{L})$ and $Sent(\mathcal{L})$.)

- If $t_1, t_2 \in Terms(\mathcal{L})$, then $= t_1 t_2 \in Wff(\mathcal{L})$.
- If $t_1, \dots, t_n \in Terms(\mathcal{L})$, then $Pt_1 \cdots t_n \in Wff(\mathcal{L})$ if $P \in Pred_n(\mathcal{L})$ or $P = X_i^n$ for some $i \in \mathbb{N}$.

2nd Order Syntax (cont.)

- If $\varphi, \psi \in \mathit{Wff}(\mathcal{L})$, then $\neg\varphi, (\varphi \to \psi) \in \mathit{Wff}(\mathcal{L})$.
- If $\varphi \in \mathit{Wff}(\mathcal{L})$, then $\forall v_i \varphi \in \mathit{Wff}(\mathcal{L})$ for any $i \in \mathbb{N}$.
- If $\varphi \in \mathit{Wff}(\mathcal{L})$, then $\forall X_i^n \varphi \in \mathit{Wff}(\mathcal{L})$ for any $i, n \in \mathbb{N}$ with $n > 0$.

So, if $\mathcal{L}_A$ is the language of PA, the following are 2nd-order sentences of $\mathcal{L}_A$.

- $\exists X_0^2\ X_0^2 \mathbf{0S0}$
- $\forall X_0^1\ (\forall v_0\ X_0^1 v_0 \to X_0^1 \mathbf{0})$
- $\exists X_0^2 \forall v_0 \forall v_1 (X_0^2 v_0 v_1 \to \neg X_0^2 v_1 v_0)$

These all "ring true" for the standard model $\mathfrak{N}$, but we need to formalize the semantics.

2nd Order Semantics

- A structure for second order logic is defined in exactly the same way as a structure for a first order language. So, the class of structures of a first order language is the same as that for a second order language having the same non-logical symbols.
- Since predicate variables occur free in various 2nd-order wffs and we want a compositional semantics, we need a way of assigning a "variable" extension to them analogous to a "sequence" s which assigns elements of the domain of discourse $|\mathfrak{A}|$ to individual variables. We simply extend the domain of s to include predicate variables s.t.
  - $s(v_i) \in |\mathfrak{A}|$ for any individual variable v_i, and
  - $s(X_i^n) \subseteq |\mathfrak{A}|^n$ for any predicate variable X_i^n.

2nd Order Semantics (cont.)

- In analogy with s^x_d, where $d \in |\mathfrak{A}|$, if $R \subseteq |\mathfrak{A}|^n$, then

$$s^{X^n_i}_R(Y) =_{df} \begin{cases} R & \text{if } Y = X^n_i \\ s(Y) & \text{otherwise.} \end{cases}$$

- Given s, we define $\overline{s}$ as follows.
  - $\overline{s}(v_i) = s(v_i)$,
  - $\overline{s}(c) = c^{\mathfrak{A}}$ if c an individual constant of $\mathcal{L}$,
  - $\overline{s}(X^n_i) = s(X^n_i)$,
  - $\overline{s}(P) = P^{\mathfrak{A}}$ if P is a predicate constant of $\mathcal{L}$, and
  - $\overline{s}(ft_1 \cdots t_n) = f^{\mathfrak{A}}(\overline{s}(t_1), \ldots, \overline{s}(t_n))$ if $t_1, \ldots, t_n \in \mathit{Terms}(\mathcal{L})$ and f is an n-ary function symbol of $\mathcal{L}$.

2nd Order Semantics (cont.)

- For $t_1, t_2 \in \mathit{Terms}(\mathcal{L})$,

$$\models_{\mathfrak{A}} = t_1 t_2 \, [s] \;\text{ iff }\; \bar{s}(t_1) = \bar{s}(t_2)$$

.

- For $t_1, \ldots, t_n \in \mathit{Terms}(\mathcal{L})$ and any n-ary predicate variable or constant Π,

$$\models_{\mathfrak{A}} \; \Pi t_1 \cdots t_n \, [s] \;\text{ iff }\; \langle \bar{s}(t_1), \ldots, \bar{s}(t_2) \rangle \in \bar{s}(\Pi).$$

- For $\varphi, \psi \in \mathit{Wff}(\mathcal{L})$,

$$\models_{\mathfrak{A}} \neg\varphi \, [s] \;\text{ iff }\; \not\models_{\mathfrak{A}} \varphi \, [s]$$

and

$$\models_{\mathfrak{A}} (\varphi \to \psi) \, [s] \;\text{ iff } \not\models_{\mathfrak{A}} \varphi \, [s] \text{ or } \models_{\mathfrak{A}} \psi \, [s].$$

2nd Order Semantics (cont.)

- For any $\varphi \in Wff(\mathcal{L})$ and any individual variable x,

$$\models_{\mathfrak{A}} \forall x \varphi\, [s] \;\text{ iff }\; \models_{\mathfrak{A}} \varphi\, [s^x_d] \text{ for all } d \in |\mathfrak{A}|.$$

- For any $\varphi \in Wff(\mathcal{L})$ and any n-ary predicate variable X,

$$\models_{\mathfrak{A}} \forall X \varphi\, [s] \;\text{ iff }\; \models_{\mathfrak{A}} \varphi\, [s^X_R] \text{ for all } R \subseteq |\mathfrak{A}|^n.$$

Lemma. If σ is a 2nd-order sentence, then $\models_{\mathfrak{A}} \sigma\, [s]$ for some s iff $\models_{\mathfrak{A}} \sigma\, [s]$ for any s.

Defn. $\models_{\mathfrak{A}} \sigma$ iff $\models_{\mathfrak{A}} \sigma\, [s]$ for some s.

2nd Order Semantics (cont.)

- Logical consequence retains its usual definition.
- So does the notion of a logical validity (entailment from the empty set).
- Examples of logical validities:
  - For any predicate constant or variable Π,

    $$\forall X \varphi \to \varphi^X_\Pi$$

    (provided Π has the same arity as X and does not get bound upon substitution if it is a variable).
  - Provided X not free in φ,

    $$\varphi \to \forall X \varphi$$

    .
  - (*Comprehension Schema*) For any n-ary predicate variable X and wff φ in which X not free,

    $$\exists X \forall x_1 \ldots \forall x_n (X x_1 \cdots x_n \leftrightarrow \varphi(x_1, \ldots, x_n)).$$

2nd Order Peano Arithmetic

Let's for present not worry about instituting logical axioms since we have a well-defined notion of logical consequence to work with. The following are *not* logical truths:

- $\forall v_0(\mathbf{0} \neq \mathbf{S}v_0)$
- $\forall v_0 \forall v_1(\mathbf{S}v_0 = \mathbf{S}v_1 \rightarrow v_0 = v_1)$
- $\forall v_0(v_0 + \mathbf{0} = v_0)$
- $\forall v_0 \forall v_1(v_0 + \mathbf{S}v_1 = \mathbf{S}(v_0 + v_1))$
- $\forall v_0(v_0 \times \mathbf{0} = \mathbf{0})$
- $\forall v_0 \forall v_1(v_0 \times \mathbf{S}v_1 = v_0 \times v_1 + v_0)$
- $\forall X_0^1([X_0^1\mathbf{0} \wedge \forall v_0(X_0^1 v_0 \rightarrow X_0^1\mathbf{S}v_0)] \rightarrow \forall v_0\ X_0^1 v_0)$

These are the axioms PA_2 of 2nd-order Peano Arithmetic.

Categoricity

Theorem. PA_2 is categorical, i.e., any two models of the axioms are isomorphic.

Pf. For any model $\mathfrak{A}$, let

$$\begin{aligned} S_A^0(\mathbf{0}^{\mathfrak{A}}) &= \mathbf{0}^{\mathfrak{A}} \\ S_A^{n+1}(\mathbf{0}^{\mathfrak{A}}) &= \mathbf{S}^{\mathfrak{A}}(S_A^n(\mathbf{0}^{\mathfrak{A}})) \end{aligned}$$

We claim that for every $d \in |\mathfrak{A}|$, $d = S_A^n(\mathbf{0}^{\mathfrak{A}})$ for some $n \in \mathbb{N}$. Let

$$\varphi(v_0) =_{df} \forall X_0^1([X_0^1\mathbf{0} \wedge \forall v_1(X_0^1 v_1 \rightarrow X_0^1\mathbf{S}v_1)] \rightarrow X_0^1 v_0).$$

By the comprehension scheme

$$\exists X_1^1 \forall v_0(X_1^1 v_0 \leftrightarrow \varphi(v_0)).$$

Categoricity (cont.)

Since $\mathfrak{A}$ is a model of PA_2 and hence of the Induction Axiom, the induction axiom must hold in $\mathfrak{A}$ for this property in particular. I.e.

$$\models_{\mathfrak{A}} [\varphi(\mathbf{0}) \wedge \forall v_0(\varphi(v_0) \rightarrow \varphi(\mathbf{S}v_0))] \rightarrow \forall v_0 \varphi(v_0).$$

Now $\varphi(\mathbf{0})$ is just the sentence

$$\forall X_0^1([X_0^1\mathbf{0} \wedge \forall v_1(X_0^1 v_1 \rightarrow X_0^1 \mathbf{S} v_1)] \rightarrow X_0^1\mathbf{0}),$$

which is a logical validity. And $\forall v_0(\varphi(v_0) \rightarrow \varphi(\mathbf{S}v_0))$ is just

$$\forall v_0\{[\forall X_0^1([X_0^1\mathbf{0} \wedge \forall v_1(X_0^1 v_1 \rightarrow X_0^1 \mathbf{S} v_1)] \rightarrow X_0^1 v_0)] \rightarrow [\forall X_0^1([X_0^1\mathbf{0} \wedge \forall v_1(X_0^1 v_1 \rightarrow X_0^1 \mathbf{S} v_1)] \rightarrow X_0^1 \mathbf{S} v_0)]\},$$

which is also a logical validity. (EXERCISE: Prove it.)

Categoricity (cont.)

Hence $\models_{\mathfrak{A}} \forall v_0 \varphi(v_0)$, i.e.,

$$\models_{\mathfrak{A}} \forall v_0 \forall X_0^1([X_0^1 \mathbf{0} \wedge \forall v_1(X_0^1 v_1 \to X_0^1 \mathbf{S} v_1)] \to X_0^1 v_0).$$

Thus, for each $d \in |\mathfrak{A}|$:

$$\models_{\mathfrak{A}} \forall X_0^1([X_0^1 \mathbf{0} \wedge \forall v_1(X_0^1 v_1 \to X_0^1 \mathbf{S} v_1)] \to X_0^1 v_0)\ [s_d^{v_0}]$$

Pick d arbitrarily and keep it fixed. Unpacking the satisfaction conditions requires in turn that

$$(*) \quad \models_{\mathfrak{A}} ([X_0^1 \mathbf{0} \wedge \forall v_1(X_0^1 v_1 \to X_0^1 \mathbf{S} v_1)] \to X_0^1 v_0)\ [(s_d^{v_0})_R^{X_0^1}]$$

for every $R \subseteq |\mathfrak{A}|$. In particular, choose

$$R = \{d \in |\mathfrak{A}| : \exists n \in \mathbb{N} \text{ s.t. } d = S_A^n(\mathbf{0}^{\mathfrak{A}})\}.$$

Categoricity (cont.)

Keep R fixed to this value. Then, since $\mathbf{0}^{\mathfrak{A}} \in R$,

$$\models_{\mathfrak{A}} X_0^1 \mathbf{0} \; [(s_d^{v_0})_R^{x_0^1}].$$

Also

$$\models_{\mathfrak{A}} \forall v_1 (X_0^1 v_1 \to X_0^1 \mathbf{S} v_1) \; [(s_d^{v_0})_R^{x_0^1}],$$

since for any $n \in \mathbb{N}$,

$$S^{\mathfrak{A}}(S_A^n(\mathbf{0}^{\mathfrak{A}})) = S_A^{n+1}(\mathbf{0}^{\mathfrak{A}})$$

and quite obviously $n + 1 \in \mathbb{N}$. It follows from $(*)$ then that

$$\models_{\mathfrak{A}} X_0^1 v_0 \; [(s_d^{v_0})_R^{x_0^1}].$$

This is true for arbitrary $d \in |\mathfrak{A}|$, so $|\mathfrak{A}| = R$, which is what we wanted to show.

Categoricity (cont.)

The final step is to show that for any other model $\mathfrak{B}$ we can establish an isomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$. The obvious mapping f is to take $S_A^n(\mathbf{0}^{\mathfrak{A}})$ to $S_B^n(\mathbf{0}^{\mathfrak{B}})$ for each $n \in \mathbb{N}$. This is obviously 1-1 and onto, and for each $d \in |\mathfrak{A}|$, $f(\mathbf{S}^{\mathfrak{A}}(d)) = \mathbf{S}^{\mathfrak{B}}(f(d))$. It remains to be shown that f is also structure preserving for addition and multiplication. This is routine. (EXERCISE) $\blacksquare$.

Cor. Cn PA_2 is complete.

Pf. If PA_2 is unsatisfiable, then PA_2 entails every sentence and is trivially complete. Otherwise, $\mathcal{K} = \text{Mod } PA_2$ is non-empty. Since $\mathcal{K}$ is an isomorphism class, for every sentence σ, either $\sigma \in \text{Th } \mathcal{K}$ or $\neg\sigma \in \text{Th } \mathcal{K}$. But Th $\mathcal{K}$ just is Cn PA_2. $\blacksquare$

Alternate Proof that Cn PA_2 Is Complete

Alternative Proof. PA_2 (if consistent) has only infinite models. Furthermore, it is $\aleph_0$-categorical. Thus, by the Los-Vaught test, Cn PA_2 is complete. ■

A Deductive Calculus for 2nd-Order Logic

The categoricity result for PA_2 is a completely semantic result. So far, we have no deductive calculus for 2nd-order logic. So let's institute such a system by adding the following as logical axioms.

- For any predicate constant or variable

$$\forall X \varphi \to \varphi^X_\Pi$$

  (provided Π has the same arity as X and does not get bound upon substitution if it is a variable).

- Provided X not free in φ,

$$\varphi \to \forall X \varphi$$

  .

A Deductive Calculus for 2nd-Order Logic (cont.)

- (*Comprehension Schema*)

$$\exists X \forall x_1 \ldots \forall x_n (X x_1 \cdots x_n \leftrightarrow \varphi(x_1, \ldots, x_n))$$

for any wff φ and n-ary predicate variable X not free in φ.

A 2nd-order derivation, like a first order derivation, is just a sequence of wffs in which every wff is either a premise, a logical axiom, or follows from previous members of the sequence by *MP*.

Incompleteness of the Deductive Calculus

- The expanded set Λ of logical axioms is such that $\#\Lambda$ is p.r.
- The axiom set A_{PA_2} for PA_2 is finite and hence is trivially p.r.
- Hence, so is the relation

$$\{\langle \#d, \#\varphi \rangle \mid d \text{ is a derivation of } \varphi \text{ from } A_{PA_2}\}.$$

- Hence this relation is representable in PA_2 by some predicate $Proof_{PA_2}(v_0, v_1)$.
- From $Proof_{PA_2}(v_0, v_1)$ one can construct along familiar lines a Gödel sentence G_2 for PA_2 s.t. if PA_2 is consistent, then $PA_2 \nvdash G_2$ and $PA_2 \nvdash \neg G_2$.
- Thus, the deductive calculus based on Λ is sound but not complete.

Incompleteness of Any P.R. Deductive Calculus

- We cannot make the deductive calculus complete by moving to some stronger Λ'. For Λ' must be decidable, and thus, by Craig's trick, $\#\Lambda'$ must be p.r., and we have the following theorem.

Theorem. Any system of 2nd-order derivation that makes

$$\{\langle \#d, \#\varphi \rangle \mid d \text{ is a derivation of } \varphi \text{ from } A_{PR_2}\}$$

p.r. is incomplete if PA_2 is consistent.

Pf. If the relation is p.r., then it is representable in PA_2 by a predicate which will allow the construction of a Gödel sentence G_2. Suppose PA_2 is consistent. Then both $PA_2 \nvdash G_2$ and $PA_2 \nvdash \neg G_2$, while at least one of them is logically entailed by PA_2. ∎

Failure of Compactness for 2nd-Order Logic

Let λ_n be the (1st-order) sentence (in "the language of equality alone") that asserts there are at least n things ($n \geq 2$), and let

$$\Sigma = \{\lambda_n \mid n \geq 2\}.$$

A 2nd-order sentence of the form

$$\exists R(\forall x \neg Rxx \wedge \forall x \forall y \forall z(Rxy \wedge Ryz \rightarrow Rxz) \wedge \forall x \exists y Rxy)$$

has all and only infinite models. (Proof: EXERCISE) Pick one of these sentences and call it λ_∞. Then every finite subset of $\Sigma; \neg\lambda_\infty$ is satisfiable, but $\Sigma; \neg\lambda_\infty$ itself is not.

Indefinability of 2nd-Order Arithmetic Truth

Theorem. If PA_2 is satisfiable, then $\#\text{Cn } PA_2$ is not definable in $\mathfrak{N}$.

Pf. The proof of the diagonalization lemma did not depend on the wffs being 1st-order. Thus the diagonalization lemma holds for PA_2, i.e., for any wff $\varphi(x)$ with only x free, there is a sentence γ s.t.

$$PA_2 \vdash \gamma \leftrightarrow \varphi(\ulcorner \gamma \urcorner).$$

Suppose $\varphi(v_0)$ defines $\#\text{Cn } PA_2$. Then there is a γ s.t.

$$PA_2 \vdash \gamma \leftrightarrow \neg\varphi(\ulcorner \gamma \urcorner),$$

and hence

$$PA_2 \models \gamma \leftrightarrow \neg\varphi(\ulcorner \gamma \urcorner).$$

Indefinability of 2nd-Order Arithmetic Truth (cont.)

Suppose PA_2 is satisfiable. Then,

$$\begin{aligned} PA_2 \models \gamma &\Leftrightarrow PA_2 \models \neg\varphi(\ulcorner\gamma\urcorner) \\ &\Leftrightarrow \models_{\mathfrak{N}} \neg\varphi(\ulcorner\gamma\urcorner) \\ &\Leftrightarrow \#\gamma \notin \#\text{Cn } PA_2 \\ &\Leftrightarrow \gamma \notin \text{Cn } PA_2 \\ &\Leftrightarrow PA_2 \not\models \gamma. \end{aligned}$$

Thus, if PA_2 is satisfiable, then $\#\text{Cn } PA_2$ is not definable in $\mathfrak{N}$. ■

Indefinability of Logical Validity

Theorem. The set of g.n.'s of 2nd-order logical validities is not definable in $\mathfrak{N}$.

Pf. Suppose otherwise. Since PA_2 is finitely axiomatized, let α be the conjunction of the axioms. We could then define $\#\mathrm{Cn}\ PA_2$ by selecting out the subset of logical validities of the form $(\alpha \to \tau)$. Explicitly, suppose that $\varphi(v_0)$ defines in $\mathfrak{N}$ the set of g.n.'s of logical validities and that $\psi(v_0)$ defines the set of g.n.'s of sentences of $\mathcal{L}_{PA}$. Let $\theta(v_0, v_1)$ define in $\mathfrak{N}$ the relation $\{\langle \#\tau, \#(\alpha \to \tau)\rangle : \models_{\mathfrak{N}} \psi(\ulcorner \tau \urcorner)\}$. Then $\exists v_1(\theta(v_0, v_1) \wedge \varphi(v_1))$ defines $\#\mathrm{Cn}\ PA_2$. ■

Contrast this with the situation in 1st-order logic, where the set of g.n.'s of logical validities is not only definable in $\mathfrak{N}$, but also effectively enumerable (because of the enumerability theorem).