HOW TO PROVE STUFF

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1. Introduction

The task in general in mathematical logic is to prove things about mathematical theories and about logic itself. The tools used are logic and mathematics, primarily set theory and arithmetic. Since some might be at sea as to how to prove something (though you know how to do proofs from geometry) here's a handy pocket guide. How to proceed generally depends on the logical (syntactic) form of the proposition to be proved.

2. Quantified Statements

These come in two varieties.

2.1. Universally quantified statements.

These are statements beginning with 'for all' (and cognates).

Examples:
- For any real number x, $x < x^2$ unless $0 \leq x \leq 1$.
- Any even integer great than 2 is the sum of two primes. (Goldbach's conjecture.)

Rule of thumb:

> Prove the statement for the case of an arbitrarily selected individual of the appropriate type, i.e., an individual (of the appropriate type) about which nothing has been previously assumed or proven.

2.2. Existentially quantified statements.

These are statements beginning with 'there exists' (and cognates).

Examples:
- There is an even prime.
- Some real numbers are less than their squares.

Rule of Thumb:

> First try constructing an individual having the property in question. If that fails, assume there is no individual with the property in question and derive a contradiction (indirect proof).

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N.B. Indirect proofs are non-constructive, and thus not acceptable in intuitionistic and constructive mathematics. Constructive proofs are to be preferred since they are universally accepted,

3. STATEMENTS WHOSE MAIN OPERATOR IS A SENTENTIAL CONNECTIVE

The most common sentential forms are: negation, conditional, conjunction, disjunction, biconditional.

3.1. **Negation.**

The canonical form is 'it is not the case that φ'.

Examples:

- $0 \neq 1$. (More canonically, it is not the case that $0 = 1$.)
- It is not the case that the ratio of circumference to diameter is rational. (Variant: The ratio of circumference to diameter is irrational.)
- No number greater than 1 is its own square. (More canonically: It is not the case that there exists a number x such that $1 < x$ and $x = x^2$.)

Rule of Thumb:

> If a direct proof is not obvious, use *reductio ad absurdum*, i.e., assume the statement that is negated and derive a contradiction.

N.B. This form of *reductio ad absurdum* is constructively acceptable.

3.2. **Conditional.**

These are statements of the form: if φ, then ψ. There are many English variants:

- If φ, ψ.
- Assuming / provided that / in the case that $/ \cdots \varphi$, then ψ.
- φ only if ψ.
- ψ if φ.

Terminology: in the above examples, φ is called the *antecedent* and ψ the *consequent*.

Rule of Thumb:

> Assume the antecedent and then derive the consequent (conditional derivation).

3.3. **Biconditional.**

These are statements of the form: φ if and only if ψ. (N.B. From now on, 'if and only if' will be abbreviatied 'iff'.)

English Variants: φ just in case ψ, φ precisely when ψ.

Rule of Thumb: Prove each of the conditionals 'if φ, then ψ' and 'if ψ, then φ'.

3.4. **Conjunction.**

Statements of the form: φ and ψ.

Rule of Thumb: You'll probably need to find proofs both of φ and of ψ.

3.5. **Disjunction.** Statements of the form: φ or ψ.

English Variant: φ unless ψ

Rule of Thumb: Assume $\neg\varphi$. Prove ψ.

(Why does this work? Here's a more verbose rendition of the line of reasoning: I want to prove 'φ or ψ'. Now, either φ or $\neg\varphi$. If φ, we're done since 'φ or ψ' follows immediately. So, suppose $\neg\varphi$. If I can prove ψ from that (and whatever other assumptions are in play), then 'φ or ψ' follows immediately. So, in either case (whether φ or $\neg\varphi$), 'φ or ψ' follows. Note that this form of proof is not intuitionistically (or constructively) acceptable, since the law of excluded middle, φ or $\neg\varphi$, is not intuitionistically valid.)

4. Some Useful Rules of Inference

Please note this is not a complete list.

4.1. ***Modus Ponens.*** This one you know. I include it just to remind you, It says, from 'if φ, then ψ' together with φ, infer ψ.

4.2. ***Modus Tollens.*** From the conditional 'if φ, then ψ' together with $\neg\psi$ infer $\neg\varphi$.

4.3. ***Modus Tollendo Ponens.*** This has two forms.

1. Given 'φ or ψ' and $\neg\varphi$, infer ψ.
2. Given 'φ or ψ' and $\neg\psi$, infer φ.

4.4. **Separation of Cases.** From the three statements 'φ or ψ', 'if φ, then χ', and 'if ψ, then χ', infer χ.

In practice, one generally has to establish the two conditionals on the fly. The rule then becomes a proof strategy with two subproofs as follows. If 'φ or ψ' is an available proposition and the goal is to prove χ, then first assume φ and show χ and second assume ψ and show χ, i.e., use conditional derivation in a pair of subproofs.

4.5. **Existential Instantiation.** This is less a rule of inference than a proof strategy. Suppose you have established or assumed an existential statement, i.e., one of the form 'there exists an x such that ... x ... ', where '$\ldots x \ldots$' is some statement about x. Then you can infer ψ if for some completely new symbol n you can derive ψ from '... n ... ' together with other established or assumed statements.

5. Cases Involving Mathematical Axioms

There are various principles assumed in number theory or set theory that underwrite proof strategies. Keep in mind that you can use all of mathematics to prove things about logic.

5.1. **Proving Set Identity.** Let A and B be sets. To show that $A = B$, show both $A \subseteq B$ and $B \subseteq A$.

This is because, according to the axiom of extensionality, sets are equal just in case they have exactly the same members.

5.2. **Proof by "Weak" Induction on the Natural Numbers $\mathbb{N}$.** To show that every natural number has the property Φ,

1. Show that 0 has the property Φ. (Base Case)
2. Show that for every natural number n, if n has Φ, then the successor of n has Φ. (Inductive Step)

5.3. **Proof by "Strong" Induction on $\mathbb{N}$.** Again this is to show that every natural number has a given property Φ: For an arbitrarily selected natural number n, show that if all $m < n$ have property Φ, then n has Φ.

Despite what might be suggested by the names, weak and strong induction on $\mathbb{N}$ are equivalent, as we will see.[1] Many students are thrown by the lack of a base case in strong induction. But the case $n = 0$ is covered since there is no natural number less than zero, so one has proved that 0 has Φ if one shows the proposition in question for arbitrary n.

Weak and strong induction on $\mathbb{N}$ are special cases of general induction principles. Weak induction is available when the set in question can be generated from a base set by various operations. In the case of the natural numbers, the base set is $\{0\}$ and there is a single operation, viz., the successor operation. Strong induction is available when there is a well-founded relation R on the set, i.e., a relation R which is such that every non-empty subset has an R-least element. In the case of $\mathbb{N}$, the relation is question is the ordinary less-than relation $<$. It is well-founded since every non-empty set of natural numbers has a $<$-least member.

A separate handout will explain the intricacies of weak and strong induction in more detail.

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[1] The reason for the terminology is that the hypothesis that one uses in the inductive step in weak induction involves only the immediately preceding number, while in strong induction it involves *all* preceding numbers.