Introductory Topics: Axiomatic Set Theory

Robert Rynasiewicz

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Formal Logical Languages in General

- What is the **syntax**?
  - What are the symbols of the language?
  - What strings of symbols count as well-formed formulas (wff's)?
  - What is a derivation of a wff from a given set of wff's?
- What is the **semantics**?
  - What is an **interpretation** of (structure for) the language?
  - What is it for a sentence σ to be **true** in an interpretation/structure? I.e., when is a structure for the language a **model** of σ?
  - What is it for a sentence to be **entailed** by, i.e., a **logical consequence** of a given set of sentences?
- Distinguish between statements in:
  - *Object Language*

    $$\text{e.g., } \forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \to x = y)$$

  - *Meta-Language*

    $$\text{e.g., } \mathrm{ZFC} \vdash \mathrm{CON}(\mathrm{ZFC}) \to \mathrm{CON}(\mathrm{ZFC} + \mathrm{CH})$$

The Language of ZF ($\mathcal{L}_{ZF}$)

- Non-logical symbols:
  $\mathcal{L}_{ZF} = \{\epsilon\}$, where ϵ is a relational predicate (binary predicate).
- Logical symbols (common to all signatures):
  $=, \neg, \rightarrow, \vee, \wedge, \leftrightarrow, \forall, \exists,$
  Individual variables: $v_0, v_1, v_2, \ldots, v_n, \ldots$

  Some of these treated as defined symbols, e.g.:
  - $\forall x$ for $\neg\exists x \neg$, or
  - $\exists x$ for $\neg\forall x \neg$.
  - $(\varphi \leftrightarrow \psi)$ for $((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$.
  - $(\varphi \wedge \psi)$ for $\neg(\varphi \rightarrow \neg\psi)$, or
  - $(\varphi \rightarrow \psi)$ for $\neg(\varphi \wedge \neg\psi)$.

  Let's adopt as primitives: $=, \neg, \rightarrow, \forall$ plus variables.

  Now we know what the symbols are. But what is a well-formed formula (wff) in ZF?

Wff's and Sentences of ZF

Inductive definition of the set of *wff's* :

1. $x \in y$ and $x = y$ are wff's for any variables x and y
2. If φ and ψ are wff's, then $\neg\varphi$ and $(\varphi \to \psi)$ are wff's.
3. If φ is a wff and x a variable, then $\forall x \varphi$ is a wff.

Nothing else is a wff. To be precise, the set of wff's is the smallest set of strings of symbols satisfying (1) – (3), where 'smallest' means w.r.t. set containment ($\subseteq$).

Recursive definition of the set of variables **free** in a wff:

- $FV(x \in y) = FV(x = y) = \{x, y\}$
- $FV(\neg\varphi) = FV(\varphi)$
- $FV(\varphi \to \psi) = FV(\varphi) \cup FV(\psi)$
- $FV(\forall x \varphi) = FV(\varphi) \setminus \{x\}$

Defn. σ is a **sentence** iff σ is a wff s.t. $FV(\sigma) = \emptyset$.

Derivations of Wff's from Sets of Wff's

Can use equivalently:

- Hilbert system (*Modus Ponens* plus logical axioms)
- Natural Deduction
- Sequent Calculus

Notation. $\Delta \vdash \varphi$ means that there is a derivation of the wff φ from the set Δ of wff's. (φ is **provable** from Δ.)

WE WILL NEVER DO A FORMAL PROOF.

Invariably, we use natural deduction *informally*, often skipping obvious steps.

Semantics

Tip: '$\mathfrak{A}$' is the letter 'A' in Euler-fraktur font.

Defn. A **structure**, or **interpretation**, $\mathfrak{A}$ for $\mathcal{L}_{ZF}$ is an ordered pair $\langle |\mathfrak{A}|, \epsilon^{\mathfrak{A}} \rangle$ where

- $|\mathfrak{A}|$ is a non-empty set (called the domain of discourse of $\mathfrak{A}$)
- $\epsilon^{\mathfrak{A}} \subseteq |\mathfrak{A}| \times |\mathfrak{A}|$, i.e., $\epsilon^{\mathfrak{A}}$ is a binary relation on $|\mathfrak{A}|$.

Let s map the set of variables to $|\mathfrak{A}|$, i.e. $s : Var \to |\mathfrak{A}|$.
Recursive definition of $\models_{\mathfrak{A}} \varphi\ [s]$:

- $\models_{\mathfrak{A}} x = y\ [s]$ iff $s(x) = s(y)$
- $\models_{\mathfrak{A}} x \in y\ [s]$ iff $\langle s(x), s(y) \rangle \in \epsilon^{\mathfrak{A}}$
- $\models_{\mathfrak{A}} \neg\varphi\ [s]$ iff $\not\models_{\mathfrak{A}} \varphi\ [s]$
- $\models_{\mathfrak{A}} (\varphi \to \psi)\ [s]$ iff $\not\models_{\mathfrak{A}} \varphi\ [s]$ or $\models_{\mathfrak{A}} \psi\ [s]$
- $\models_{\mathfrak{A}} \forall x\varphi\ [s]$ iff $\models_{\mathfrak{A}} \varphi\ [s^x_d]$ for all $d \in |\mathfrak{A}|$, where $s^x_d(x) = d$ but is the same as s on all other variables.

Semantics (cont.)

Lemma. If $FV(\sigma) = \emptyset$, then $\models_{\mathfrak{A}} \sigma\ [s]$ for some s iff $\models_{\mathfrak{A}} \sigma\ [s]$ for any s.

Suppose τ is a sentence and Σ is a set of sentences.

Defn. $\models_{\mathfrak{A}} \tau$ iff $\models_{\mathfrak{A}} \tau\ [s]$ for some s.

Terminology. We say that

- $\mathfrak{A}$ is a **model** of τ iff $\models_{\mathfrak{A}} \tau$. (Variants: $\mathfrak{A}$ **satisfies** τ, τ is true in $\mathfrak{A}$)
- $\mathfrak{A}$ is a **model** of Σ iff $\mathfrak{A}$ is a model of τ for each $\tau \in \Sigma$.

Defn. $\Sigma \models \tau$ iff every model of Σ is a model of τ.

Defn. Σ is **satisfiable** iff Σ has a model.

Defn. τ is **independent** of Σ iff neither $\Sigma \models \tau$ nor $\Sigma \models \neg\tau$.

Relation between Syntax and Semantics

Theorems:

- **Deductive Soundness**: $\Sigma \vdash \tau \Rightarrow \Sigma \models \tau$.
  Equivalently, any satisfiable set of sentences is consistent.
- **Deductive Completeness**: $\Sigma \models \tau \Rightarrow \Sigma \vdash \tau$.
  Equivalently, any consistent set of sentences is satisfiable.

Defn. τ is independent of Σ iff neither $\Sigma \vdash \tau$ nor $\Sigma \vdash \neg\tau$.

Defn. $\mathrm{Con}(\Sigma)$ iff there is no β s.t. $\Sigma \vdash \beta$ and $\Sigma \vdash \neg\beta$.

Comment. If Σ is inconsistent, then Σ entails every sentence of $\mathcal{L}_{ZF}$.

Lemma. $\Sigma \nvdash \tau$ iff $\mathrm{Con}(\Sigma \cup \{\neg\tau\})$.

Corollary. τ is independent of Σ iff $\mathrm{Con}(\Sigma \cup \{\neg\tau\})$ and $\mathrm{Con}(\Sigma \cup \{\tau\})$.

Pragmatics

Unqualified advice:

- To show that $\Sigma \vdash \tau$, give a proof of τ from Σ.
- To show that $\Sigma \nvdash \tau$, find a model of $\Sigma \cup \{\neg\tau\}$. This is equivalent to showing that $\Sigma \cup \{\neg\tau\}$ is consistent.

Qualified advice: In some contexts,

- to show $\mathrm{Con}(T)$, show that T has a model, and
- to show that $\mathrm{Con}(T) \rightarrow \mathrm{Con}(T')$, show that *if* T has a model, then T' has a model.

The reason for qualification is that it may be possible to show that T has a model if one uses resources *stronger than* the theory T. For example, let T be Peano Arithmetic (see next slide). It can be shown that T has a model **if** one assumes there are infinite sets. But that is not an assumption of Peano Arithmetic. So, one can't show it consistent by producing, say, the standard model. In fact, no one knows whether Peano Arithmetic is consistent, absolutely speaking.

Axioms of Peano Arithmetic

Note: S is intended to be the successor function on $\mathbb{N}$, i.e., the successor of n is $n + 1$.

- $\forall x : 0 \neq Sx$
- $\forall x \forall y (Sx = Sy \rightarrow x = y)$
- $\forall x : x + 0 = x$
- $\forall x \forall y (x + Sy = S(x + y)$
- $\forall x : x \cdot 0 = 0$
- $\forall x \forall y (x \cdot Sy) = x \cdot y + x$
- (Induction Principle) For any wff φ, if $FV(\varphi) = \{x, z_1, \ldots, z_n\}$, then the following is an axiom

  $$\forall z_1, \ldots, z_n ((\varphi(0) \wedge \forall x (\varphi(x) \rightarrow \varphi(Sx)) \rightarrow \forall x \varphi(x)),$$

  where $\varphi(t)$ is the result of uniformly substituting t for x in φ.

Now for the axioms of ZF!

Extensionality

Axiom of Extensionality: $\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y)$

Notation. $x \subseteq y$ stands for $\forall z (z \in x \rightarrow z \in y)$

Then extensionality is equivalent to:

$$\forall x \forall y ((x \subseteq y \wedge y \subseteq x) \rightarrow x = y)$$

Standard procedure for proving sets A and B are the same:

- Show $A \subseteq B$
- Show $B \subseteq A$
- Conclude $A = B$

Remark on notation: Kunen uses $\subset$ for $\subseteq$. Some authors use $A \subset B$ for $(A \subseteq B \wedge A \neq B)$, i.e., to say that A is a *proper* subset of B.

Extensionality (cont.)

Remark 1: There is no predicate for ‘is a set’ in ZF. ZF presupposes everything is a set, or rather, ZF is only about sets, or at least from the point of view of the object language.

This is to say, axiomatic set theory is about *hereditary* sets, i.e., sets whose members are sets, whose members are sets, and so on.

If we were to include UR-elements (individuals that are not sets), then extensionality would fail and we would have to introduce a predicate *Set* in the formal language and invoke:

$$\forall x \forall y (Set(x) \wedge Set(y) \rightarrow (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y))$$

Interpretations

Remark 2: From the point of view of the meta-language:

- Some interpretations $\mathfrak{A} = \langle |\mathfrak{A}|, \epsilon^{\mathfrak{A}} \rangle$ are such that $|\mathfrak{A}|$ consists of entirely of sets and $\epsilon^{\mathfrak{A}}$ is the set membership relation on these sets. In this case, we speak (from the point of view of the meta-language) of *hereditary* sets.
- Some interpretations are such that $\epsilon^{\mathfrak{A}}$ is the set membership relation on $|\mathfrak{A}|$, but not all items in $|\mathfrak{A}|$ are sets. These are interpretations using sets with *UR-elements*. Extensionality will typically fail in these interpretations.
- Some interpretations are such that neither $|\mathfrak{A}|$ consists of sets nor $\epsilon^{\mathfrak{A}}$ is the set membership relation. These are *non-standard* interpretations. Extensionality may or may not be satisfied.

Extensionality (cont.)

Directed Graph of a relation R on a set A: Set of directed edges and vertices such that

- Each element of A is a vertex.
- For each $\langle a, b \rangle \in R$ there is a directed edge from vertex a to vertex b.

Examples to be drawn:

1. Successor relation on $\mathbb{N}$.
2. Three element divergent partial ordering.
3. Three element divergent strict partial ordering.
4. Three element convergent partial ordering.
5. Three element convergent strict partial ordering.

Which of the above satisfy the axiom of extensionality?

Converse of Extensionality

- What about the "converse" of extensionality?

$$\forall x \forall y (x = y \to \forall z (z \in x \leftrightarrow z \in y))$$

- Leibniz's Law: If $t = t'$, then, for any predicate $\varphi(x)$, if $\varphi(t)$, then $\varphi(t')$.
- $\vdash \forall x \forall y (x = y \to \forall z (z \in x \leftrightarrow z \in y))$ by (iterated) use of Leibniz Law.

Axiom of Foundation

Notation.

- $(\forall x \in y)\ \varphi$ is short for $\forall x(x \in y \to \varphi)$
- $(\exists x \in y)\ \varphi$ is short for $\exists x(x \in y \land \varphi)$

Axiom of Foundation. $\forall x(\exists y(y \in x) \to \exists y \in x\ \forall z(z \in x \to z \notin y))$
Paraphrases:

- Every non-empty set x has an element disjoint with x.
- Every non-empty set has an ϵ-minimal element.

Foundation rules out such "pathological" sets as those that constitute naked one-cycles, i.e., sets whose only members are themselves:

Proposition. Found $\vdash \neg\exists x \forall y(y \in x \leftrightarrow y = x)$.
Pf. Suppose contrary to what is to be shown that such a set x exists. Clearly $\exists y(y \in x)$, namely x itself. Hence there must be an element $y \in x$ s.t. $(\forall z \in x)(z \notin y)$. But the only element of x is x itself. So, x must be such that $(\forall z \in x)(z \notin x)$, which can be true only if x has no members. ■

Axiom of Foundation (cont.)

Ultimately, foundation will be use to rule out other "pathological" cases such as the following.

- Closed cycles: $x_0 \in x_1 \in \cdots \in x_n \in x_0$
- Infinite regresses: $\cdots \in x_n \in \cdots \in x_2 \in x_1 \in x_0$

But in order to do so, certain other of the axioms of ZF are needed. In fact, at this point, we cannot even rule out such "clothed" one cycles as in the following structure $\mathfrak{A}$:

- $|\mathfrak{A}| = \{a, b\}$
- $\epsilon^{\mathfrak{A}} = \{\langle a, b\rangle, \langle b, b\rangle\}$

N.B. Let ZF^- be ZF w/o Foundation. Whether or not there might be "pathological" sets as the ones mentioned isn't important. We will argue eventually that the existence of all standard mathematical objects is guaranteed by ZF^-. That is, in any model of ZF^-, there is a sub-model containing all standard mathematics and satisfying ZF.

Comprehension Schema

The intuitive idea:

- Suppose z is some set.
- Let P be some set-theoretic property.
- The collection of all elements of z that have the property P then forms a set (which is obviously then a subset of z).

Defn. Suppose $FV(\varphi) = \{w_1, \ldots, w_n\}$. Then the *universal closure* of φ is $\forall w_1 \cdots \forall w_n \varphi$.

Comprehension Schema: Let x, y, z be distinct variables and let φ be any wff s.t. $y \notin FV(\varphi)$. Then the universal closure of

$$\forall z \exists y \forall x (x \in y \leftrightarrow (x \in z \wedge \varphi))$$

is an axiom.

The Comprehension Schema is often called the **Axiom of Separation**.

Comprehension together with Extensionality

If we assume both Extensionality and the Comprehension Schema, then we have an infinite number of theorems of the following form.

Theorem Schema. Let x, y, z be distinct variables and let φ be any wff s.t. $y \notin FV(\varphi)$. Then the universal closure of

$$\forall z \exists ! y \forall x (x \in y \leftrightarrow (x \in z \wedge \varphi))$$

is provable from Ext+Compr.

Scholium. For a given z and φ, one is tempted to write

$$\{x \in z \mid \varphi\}$$

for the unique y guaranteed to exists by the Comprehension Schema. But there's a **puzzle** as to what this stands for in the object language, which we will put off for awhile.

Non-existence of a Universal Set

Theorem. Compr $\vdash \neg\exists x \forall y (y \in x)$.
Pf. Suppose such a set U exists. By the Comprehension Schema, the following is an axiom.

$$\forall z \exists y \forall x (x \in y \leftrightarrow (x \in z \wedge x \notin x)).$$

Hence, by universal instantiation,

$$\exists y \forall x (x \in y \leftrightarrow (x \in U \wedge x \notin x)).$$

Let C be such a y, so that

$$\forall x (x \in C \leftrightarrow (x \in U \wedge x \notin x)).$$

By universal instantiation,

$$(C \in C \leftrightarrow (C \in U \wedge C \notin C)).$$

Non-existence of a Universal Set(cont.)

Since by hypotheses $\forall y (y \in U)$, we have $C \in U$ in particular. So, it follows that

$$C \in C \leftrightarrow C \notin C,$$

which is contradictory. ■

Existence of Empty Set

Theorem. Ext + Compr $\vdash \exists! y \forall x (x \notin y)$.
Pf. Existence: Let φ be the wff $x \neq x$. Then

$$\forall z \exists y \forall x (x \in y \leftrightarrow (x \in z \wedge x \neq x))$$

is an axiom. Since by set existence $\exists x (x = x)$, let c be one such x. Then, universally instantiating the above instance of the Comprehension Schema,

$$\exists y \forall x (x \in y \leftrightarrow (x \in c \wedge x \neq x)).$$

Let d be such a y. So

$$\forall x (x \in d \leftrightarrow (x \in c \wedge x \neq x)).$$

We want to show that d is such that $\forall x (x \notin d)$.

Existence of Empty Set (cont.)

Pick x arbitrarily and suppose that $x \in d$. From the last displayed line it follows *inter alia* that $x \neq x$, which contradicts the logical validity $x = x$. It follows that $x \notin d$. By universal generalization, $\forall x (x \notin d)$. By existential generalization $\exists y \forall x (x \notin y)$.

Uniqueness: More informally (but still in the object language), suppose d and d' are both such that they have no members. Then they have exactly the same members. So, by Extensionality, $d = d'$. ■

Meta-Theorem. No other set can be proved to exist from Ext + Compr alone. I.e. Con(Ext+Compr + $\exists x \forall y \; y = x$).
Pf. Let $|\mathfrak{A}|$ contain a single element and let $\epsilon^{\mathfrak{A}} = \emptyset$. Then $\mathfrak{A}$ satisfies both Ext and Compr.

FAQ's about Comprehension

1. Why allow other variables in addition to x to be free in φ?
2. Why the prohibition on $y \notin FV(\varphi)$?
3. How do I tell if an arbitrarily given interpretation $\mathfrak{A}$ of $\mathcal{L}_{ZF}$ satisfies all instances of Comprehension (since there are so many instances)?
4. Why have you been avoiding using set-theoretic notation such as: $\emptyset$, $\{x\}$, $\{x_1, \ldots, x_n\}$, $\{x \in z \mid \varphi(x)\}$, etc.?

Why allow other variables in addition to x to be free in φ?

Although the only set you can prove to exist (absolutely) with Extensionality and Comprehension is the empty set, we can still show that some sets exist provided others exist.

E.g., let φ in the Comprehension Schema be the wff $x \in w$. This yields the axiom:

$$\forall w \forall z \exists y \forall x (x \in y \leftrightarrow (x \in z \land x \in w)).$$

So, in any model $\mathfrak{A}$ of Comprehension the intersection of any two sets must exist. Also the intersection of arbitrarily finitely many sets for arbitrary $n + 1$:

$$\forall w_1 \cdots \forall w_n \forall z \exists y \forall x (x \in y \leftrightarrow (x \in z \land x \in w_1 \land \cdots \land x \in w_n)).$$

N.B. Set differences must exist (let φ be $x \notin w$).
N.B. But not unions! (Why not?)

EXERCISE

Problem. Show that

$$\text{Ext} + \text{Compr} \nvdash \forall u \forall v \exists y \forall x (x \in y \leftrightarrow (x \in u \vee x \in v)).$$

I.e., show that it might be the case that sets A and B exist, but $A \cup B$ does not, even if Extensionality and Comprehension are true.

Solution

Let

- Let $|\mathfrak{A}| = \{a, b, c\}$ and
- $\epsilon^{\mathfrak{A}} = \{\langle a, b\rangle, \langle b, c\rangle\}$

N.B. Foundation is also satisfied.

Why the prohibition on $y \notin FV(\varphi)$?

This would permit "self-reference" in the defining condition φ, which would severely hamper the further development of the theory. E.g., let φ be $x \notin y$. Then we would have:

$$\forall z \exists y \forall x (x \in y \leftrightarrow (x \in z \wedge x \notin y))$$

as an axiom. Let C be an arbitrary set. Then

$$\exists y \forall x (x \in y \leftrightarrow (x \in C \wedge x \notin y)).$$

So, let D be such a y. Thus,

$$\forall x (x \in D \leftrightarrow (x \in C \wedge x \notin D)).$$

Now, suppose $a \in C$, i.e., that C is non-empty. Then $a \in D$ iff $a \notin D$. This leads to an explicit contradiction. Hence C must be empty. Thus, Ext+Compr would have at best a single model $\mathfrak{A}$ up to isomorphism such that $|\mathfrak{A}|$ is a singleton set and $\epsilon^{\mathfrak{A}} = \emptyset$.

How do I tell if an arbitrarily given interpretation $\mathfrak{A}$ of $\mathcal{L}_{ZF}$ satisfies all instances of Comprehension?

Problem: There is an axiom for each wff φ. How do you check through all the wff's?

Answer: You can't. But you can get some help from knowing what subsets of $|\mathfrak{A}|$ and what relations on $|\mathfrak{A}|$ are definable.

Notation. Suppose $FV(\varphi) = \{v_0, \dots, v_n\}$ and $a_0, \dots, a_n \in |\mathfrak{A}|$. Then

$$\models_{\mathfrak{A}} \varphi \, [\![a_0, \dots, a_n]\!]$$

iff $\models_{\mathfrak{A}} \varphi \, [s]$ for some $s : Var \to |\mathfrak{A}|$ s.t. $s(v_i) = a_i$ for $0 \leq i \leq n$.

Defn. Let $R \subseteq |\mathfrak{A}|^{n+1}$. Then R is **definable in** $\mathfrak{A}$ iff there is a wff φ such that $FV(\varphi) = \{v_0, \dots, v_n\}$ and

$$R = \{\langle a_0, \dots, a_n \rangle \mid \, \models_{\mathfrak{A}} \varphi \, [\![a_0, \dots, a_n]\!]\}.$$

How do I tell if an arbitrarily given interpretation $\mathfrak{A}$ of $\mathcal{L}_{ZF}$ satisfies all instances of Comprehension? (cont.)

Ad Hoc Notation. Suppose $b \in |\mathfrak{A}|$. Let

$$Pred(b) = \{c \in |\mathfrak{A}| \colon \langle c, b\rangle \in \epsilon^{\mathfrak{A}}\}.$$

Thus, $Pred(b)$ is the set of all elements of the domain of discourse that qualify as members of b according to $\mathfrak{A}$.

Method: For each $b \in |\mathfrak{A}|$, check to see if for each definable subset C of $Pred(b)$ there is $d \in |\mathfrak{A}|$ such that $C = Pred(d)$.

Automorphism Theorem. A subset of $|\mathfrak{A}|$ or a relation on $|\mathfrak{A}|$ is definable in $\mathfrak{A}$ *only if* it is preserved under all automorphisms.

Unfortunately, this gives only a *necessary* condition for definability in $\mathfrak{A}$, and so only helps to tell us when certain relations are *not* definable in $\mathfrak{A}$. In general, the more symmetric the structure, the fewer the definable relations in $\mathfrak{A}$, and the "easier" it is to satisfy Compr.

How do I tell if an arbitrarily given interpretation $\mathfrak{A}$ of $\mathcal{L}_{ZF}$ satisfies all instances of Comprehension? (cont.)

1. Example in which Comp is satisfied:
    - $|\mathfrak{A}| = \{a, b, c, d\}$
    - $\epsilon^{\mathfrak{A}} = \{\langle a, b\rangle, \langle a, c\rangle, \langle b, d\rangle, \langle c, d\rangle\}$

   Note that $\mathfrak{A}$ is symmetric under interchange of b and c, so neither $\{b\}$ nor $\{c\}$ is definable.
2. Example in which Comp fails:
    - $|\mathfrak{B}| = \{a, b, c\}$
    - $\epsilon^{\mathfrak{B}} = \{\langle a, b\rangle, \langle a, c\rangle, \langle b, c\rangle\}$

   Let φ in the Comprehension Schema be $\exists! u\ u \in x$. Then

   $$\not\models_{\mathfrak{B}} \forall z \exists y \forall x (x \in y \leftrightarrow (x \in z \wedge \exists! u\ u \in x)).$$

   For $\exists! u\ u \in x$ defines $\{b\} \subseteq |\mathfrak{B}|$, but there is no $d \in |\mathfrak{B}|$ s.t. $Pred(d) = \{b\}$.

An Application to Foundation

Foundation together with the Comprehension Schema rules out the existence of 1-cycles, i.e.,

Proposition. Found+Compr $\vdash \forall x\, x \notin x$.

Pf. Suppose there is some x s.t. $x \in x$. Call it a. Consider the instance of the Comprehension Schema having $x = z$ for φ:

$$\forall z \exists y \forall x (x \in y \leftrightarrow (x \in z \wedge x = z)).$$

Instantiating with a:

$$\exists y \forall x (x \in y \leftrightarrow (x \in a \wedge x = a)).$$

Let b be such a y. Thus,

$$\forall x (x \in b \leftrightarrow (x \in a \wedge x = a)).$$

Since by hypothesis $a \in a$ and it's a logical truth that $a = a$, we have that $a \in b$. Conversely, if $x \in b$, then $x = a$. So $x \in b$ iff $x = a$. That is, $b = \{a\}$. But then b violates foundation, since b is non-empty, and its only element a is such that it has a member in common with b, namely a itself. ■

Defined Symbols

The above proof was carried out in the object language, but in addition to the epsilon symbol we used bracket notation. We'd also like to use $\emptyset$ for the empty set when reasoning in the object language. What's the general policy regarding defined notation?

- For any wff $\varphi(x_1, \ldots, x_n)$ we can introduce a metalinguistic abbreviation $Px_1 \cdots x_n$ for it w/o further ado with the understanding that
$$\forall x_1 \cdots \forall x_n (Px_1 \cdots x_n \leftrightarrow \varphi(x_1 \cdots x_n)).$$
  E.g. $x \subseteq y$ for $\forall z(z \in x \to z \in y)$.
- But introducing a defined constant or function symbol requires first proving existence and uniqueness results. And then saying what they abbreviate in the object language is trickier.

So as to have more existence results at hand, let's first introduce the Pairing Axiom, or Axiom of Unordered Pairs.

Pairing, a.k.a. Unordered Pairs

Axiom of Pairing. $\forall x \forall y \exists z (x \in z \wedge y \in z)$.

Theorem. Ext+Compr+UP $\vdash \forall u \forall v \exists! y \forall x (x \in y \leftrightarrow (x = u \vee x = v))$.

Pf. Let u and v be any two sets. By UP, there exists a z s.t. $u \in z$ and $v \in z$. By Ext+Compr, with the wff $(x = u \vee x = v)$ for φ, we get

$$\exists! y \forall x (x \in y \leftrightarrow (x \in z \wedge (x = u \vee x = v))).$$

By universal generalization we get the conclusion. ■

We'd like to be able to say that the upshot of this theorem is that, for any sets x and y, there exists the set $\{x, y\}$.

But what does $\{x, y\}$ stand for in the object language?

Defined Function Symbols and Constants

- First, consider the simpler case of $\{x\}$.
- Second, think of the braces as a function symbol f, i.e. $fx = \{x\}$.
- Third, think of the necessary and sufficient conditions for $z = fx$, in this case $\forall y(y \in z \leftrightarrow y = x)$.
- Fourth, we need a guarantee that for each x such a z exists and is unique:
  $\mathrm{Ext} + \mathrm{UP} + \mathrm{Compr} \vdash \forall x \exists ! z \forall y(y \in z \leftrightarrow y = x)$.
- Then the statement, e.g., $u \in fx$, i.e., $u \in \{x\}$, is shorthand for:
  $\exists ! z(\forall y(y \in z \leftrightarrow y = x) \wedge u \in z)$.
- For an arbitrary wff $\psi(\cdot)$, the expression $\psi(\{x\})$ is shorthand for:
  $\exists ! z(\forall y(y \in z \leftrightarrow y = x) \wedge \psi(z))$.

Defined Function Symbols and Constants (cont.)

- The metalinguistic abbreviation $\psi(\{x_1, x_2\})$ is short for: $\exists! z(\forall y(y \in z \leftrightarrow (y = x_1 \vee y = x_2)) \wedge \psi(z))$
- $\{x_1, \ldots, x_n\}$ just behaves as an n-ary function symbol with the obvious existence and uniqueness conditions.
- The notation $\{x \mid \varphi\}$ is treated as a metalinguistic abbreviation thusly. The wff $\psi(\{x \mid \varphi(x)\})$ is short for: $\exists! z(\forall y(y \in z \leftrightarrow \varphi(y)) \wedge \psi(z))$.
- $\{x \in w \mid \varphi\}$ is just $\{x \mid x \in w \wedge \varphi\}$.
- A defined constant (name) is treated as a 0-ary function symbol. E.g., $\psi(\emptyset)$ is short for: $\exists! z(\forall y(y \notin z) \wedge \psi(z))$.
- So, for any wff $\theta(z)$ with one free variable z, if

$$\text{Axioms} \vdash \exists! z\theta(z),$$

then we can introduce an individual constant c s.t. for any wff $\psi(x)$, $\psi(c)$ is short for: $\exists! z(\theta(z) \wedge \psi(z))$.

Pairing and Foundation

With Pairing (and the Comprehension Schema) Foundation now rules out 2-cycles, i.e.,

Proposition. Found + Compr + UP $\vdash \neg\exists x \exists y (x \in y \ \wedge \ y \in x)$.

Pf. Suppose x and y are s.t. $x \in y$ and $y \in x$. By Pairing and Comprehension, there is a two element set A with exactly x and y as members. (I refrain from saying that $\{x, y\}$ exists, for that presupposes Extensionality, which we don't need.) Since

- $y \in A \cap x$,
- $x \in A \cap y$, and
- x and y are the only members of A,

A has no member disjoint with A. In other words, A violates Foundation.

Models of the Pairing Axiom

Claim. Pairing by itself has finite models.
Trivially, take $|\mathfrak{A}| = \{a\}$ and $\epsilon^{\mathfrak{A}} = \{\langle a, a\rangle\}$.

Claim. Ext+Compr+UP has no finite models.
Pf. Suppose $\mathfrak{A}$ is a finite model. Then for some natural number n, $|\mathfrak{A}|$ has exactly $n+1$ distinct elements $a_0, \ldots, a_n$. According to UP and Ext, $|\mathfrak{A}|$ has $n+1$ distinct elements $b_0, \ldots, b_n$ s.t. $\langle a_i, b_j\rangle \in \epsilon^{\mathfrak{A}}$ iff $i = j$ for $0 \leq i, j \leq n$. Since $|\mathfrak{A}|$ has exactly $n+1$ elements, for each j there exists a unique i s.t. $b_j = a_i$. Then, since each b_j is non-empty, so is each a_i. But, since $\mathfrak{A}$ satisfies Ext and Compr, one of the a_i's must be empty, which is a contradiction. ■

Pairing Axiom (cont.)

Claim. Ext+Compr+UP, if it has a model, has a model in which no set has more than two members.

Pf. Let $\mathfrak{A}$ be a model. We construct a submodel $\mathfrak{C}$ as follows.

(1) Let $\emptyset^{\mathfrak{A}}$ be the unique element $a \in |\mathfrak{A}|$ for which there is no $b \in |\mathfrak{A}|$ such that $\langle b, a\rangle \in \epsilon^{\mathfrak{A}}$.

(2) Let $g : |\mathfrak{A}| \times |\mathfrak{A}| \to |\mathfrak{A}|$ be s.t. for all $a, b \in |\mathfrak{A}|$, $g(a, b)$ is the unique $c \in |\mathfrak{A}|$ such that $\langle d, c\rangle \in \epsilon^{\mathfrak{A}}$ iff $d = a$ or $d = b$.

(3) By a *construction sequence* on $|\mathfrak{A}|$, we mean a finite sequence $s_0, \ldots, s_n$ of elements of $|\mathfrak{A}|$ s.t. for each $k \leq n$ either

(i) $s_k = \emptyset^{\mathfrak{A}}$, or
(ii) $s_k = g(s_i, s_j)$ for $i, j < k$.

(4) For each $m \in \mathbb{N}$, let

$$C_m =_{df} \{c_m \,|\, \text{there exists a construction sequence } c_0, \ldots, c_m\}.$$

Proof (cont.)

(5) Now let

$$C = \bigcup \{ C_n \mid n \in \mathbb{N} \}.$$

(6) Finally, let $\mathfrak{C} = \langle C, \epsilon^{\mathfrak{A}} \restriction C \rangle$, where $\epsilon^{\mathfrak{A}} \restriction C$ is the restriction of $\epsilon^{\mathfrak{A}}$ to C.

It follows by induction on $\mathbb{N}$ that both Ext and Compr are satisfied in each $\langle C_n, \epsilon^{\mathfrak{A}} \restriction C_n \rangle$.

It is immediate then that they are both satisfied in $\mathfrak{C}$ (because otherwise they would fail in some $\langle C_n, \epsilon^{\mathfrak{A}} \restriction C_n \rangle$).

And UP is satisfied in $\mathfrak{C}$ since C is closed under g. ■

A Corollary

Corollary. Set union still cannot be defined.

Pf. Since C is infinite, we can find distinct members $a, b, c \in C$. If the union of $g(a, a)$ with $g(b, c)$ existed in C, it would have three elements. But no set in C has more than two elements. $\blacksquare$

Existence of 3-Cycles

Recall that UP + Compr + Found rules out 2-cycles, since for any x and y that might form a 2-cycle, we can form the two element set $\{x, y\}$ which is then not well-founded. That we can't prove that there are three element sets, however, suggests that we won't be able to rule out 3-cycles.

Claim. UP + Compr + Ext + Found does not rule out 3-cycles, assuming it does not rule out everything.

Pf. Expand the model $\mathfrak{C} = \langle C, \epsilon^{\mathfrak{A}} \restriction C \rangle$ from the last proof to a structure $\mathfrak{B}$ as follows. Suppose $a, b, c \notin C$ are distinct. Let $E = \{a, b, c\}$ and $R = \{\langle a, b\rangle, \langle b, c\rangle, \langle c, a\rangle\}$. We now mean by a construction sequence on $|\mathfrak{A}| \cup E$ a finite sequence s.t. for each $k \leq n$ either
(i) $s_k \in \{\emptyset^{\mathfrak{A}}, a, b, c\}$ or
(ii) $s_k = g(s_i, s_j)$ for $i, j < k$.

Next, for each $m \in \mathbb{N}$, let

$$B_m =_{df} \{b_m \,|\, \text{there exists a construction sequence } b_0, \ldots, b_m\}$$

and $B = \bigcup\{B_n \,|\, n \in \mathbb{N}\}$.

Existence of 3-Cycles (cont.)

Next, define

$$\begin{array}{rcl} S_0 & = & R \cup \epsilon^{\mathfrak{A}} \restriction C \\ S_{n+1} & = & S_n \cup \{\langle x, g(x,y)\rangle \mid x, y \in B_n\} \\ S & = & \bigcup\{S_n \mid n \in \mathbb{N}\}. \end{array}$$

Then let $|\mathfrak{B}| = B$ and $\epsilon^{\mathfrak{C}} = S$.

By induction on n, Foundation is satisfied for each element $x \in B_n$ for each $n \in \mathbb{N}$. Thus, it is satisfied throughout B. That Ext is satisfied is clear, again by induction on n. UP is satisfied by force in closing $C \cup E$ under unordered pairing. As for Compr, each element of B_0 satisfies it. And if B_n satisfies it, so does B_{n+1}, since only two-element sets are generated, each of whose 1-element subsets are also generated. Yet

$$\models_{\mathfrak{E}} (x \in y \wedge y \in z \wedge z \in x)\ [\![a, b, c]\!],$$

So

$$\models_{\mathfrak{E}} \exists x \exists y \exists z (x \in y \wedge y \in z \wedge z \in x).\blacksquare$$

Ordered Pairs, Relations, Functions Defined

Defn.:

$$\langle x, y\rangle =_{df} \{\{x\},\{x,y\}\}.$$

Lemma.
Ext + UP + Compr $\vdash \forall x \forall y \forall u \forall v(\langle x, y\rangle = \langle u, v\rangle \leftrightarrow (x = u \ \wedge \ y = v))$.

Pf. Exercise.

Defn. A **relation** is a set of ordered pairs, i.e.,

$$Rel(R) =_{df} \forall y \in R\, \exists u \exists v(y = \langle u, v\rangle).$$

Defn. A **function** is a set of single valued ordered pairs, i.e.,

$$Fn(f) =_{df} Rel(f) \wedge \forall x \forall y_1 \forall y_2((\langle x, y_1\rangle \in f \wedge \langle x, y_2\rangle \in f) \rightarrow y_1 = y_2).$$

Ordered Pairs, Relations, Functions Defined (cont.)

Notation. If $Fn(f)$ and $\exists y(\langle x, y\rangle \in f)$, then $\mathbf{f(x)}$ is the unique y s.t. $\langle x, y\rangle \in f$.

N.B. If $Rel(R)$, it is standard in mathematics to define

$$dom(R) =_{def} \{x \mid \exists y(\langle x, y\rangle \in R)\}$$

and

$$ran(R) =_{def} \{y \mid \exists x(\langle x, y\rangle \in R)\}.$$

Is it guaranteed that $dom(R)$ and $ran(R)$ exist for every relation R?

NO!

Axiom of Union

Axiom of Union. $\forall x \exists y \forall z (\exists u \in x (z \in u) \to z \in y)$.

Equivalently: $\forall x \exists y \forall u \forall z ((z \in u \wedge u \in x) \to z \in y)$.

Example.

Let $\mathscr{F} = \{A, B, C\}$; $A = \{a_1, a_2\}$; $B = \{b\}$; $C = \{c, a_1\}$.
Then there is a set S such that $a_1 \in S$, $a_2 \in S$, $b \in S$, and $c \in S$.

Furthermore, by Compr + Ext, there is a unique $D = \{a_1, a_2, b, c\}$.

Generalizing, for arbitrary $\mathscr{F}$:

$$\bigcup \mathscr{F} =_{df} \{x \mid \exists Y (Y \in \mathscr{F} \ \wedge \ x \in Y)\}.$$

Also, if $\mathscr{F} \neq \emptyset$:

$$\bigcap \mathscr{F} =_{df} \{x \in \bigcup \mathscr{F} \mid \forall Y (Y \in \mathscr{F} \to x \in Y)\}.$$

Axiom of Union (cont.)

Definitions:

$x \cup y =_{df} \bigcup\{x, y\}$

$S(x) =_{df} x \cup \{x\}$

Now we define:

$$\begin{aligned} 0 &=_{df} \emptyset \\ n+1 &=_{df} S(n) \end{aligned}$$

I.e.,
$0 = \emptyset$
$1 = \{\emptyset\} = \{0\}$
$2 = \{\emptyset, \{\emptyset\}\} = \{0, 1\}$
$3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \{0, 1, 2\}$
$\vdots$
N.B.: $0 \in 1 \in 2 \in 3 \in \cdots$ [transitively].

A Lemma Schema

Lemma Schema. For each $n \in \mathbb{N}$ the following is provable from Ext + Compr + UP + U:

$$\forall x_0 \cdots \forall x_n \exists ! y \forall z (z \in y \leftrightarrow (z = x_0 \ \vee \cdots \vee \ z = x_n)).$$

Pf. By induction on n. The case of n=0, i.e., of singleton sets, was already proven from Compr + UP. So assume the schema for n is provable. We show the schema for $n+1$ is provable. Given $x_0, \ldots, x_n$, by hypothesis $\{x_0, \ldots, x_n\}$ exists. Given x_{n+1}, by UP $\{x_{n+1}\}$ exists. Applying UP again, $\{\{x_0, \ldots, x_n\}, \{x_{n+1}\}\}$ exists. By U, we can take the union over this to get $\{x_0, \ldots, x_{n+1}\}$. ■

Application to Foundation: There are no *n*-cycles.

Corollary Schema. For any $n \in \mathbb{N}$, the following is derivable from Found + Compr + UP +U:

$$\neg \exists x_0 \cdots \exists x_n (x_0 \in x_1 \in \cdots \in x_n \in x_0).$$

Pf. For any n, suppose that $x_0, \ldots, x_n$ are s.t.

$$x_0 \in x_1 \in \cdots \in x_n \in x_0.$$

By the lemma, we have as a set $A = \{x_0, \ldots, x_n\}$. Since A is non-empty, A must contain some element x_i disjoint with A. If $i = 0$, then $x_n \in A \cap x_i$. If $i > 0$, then $x_{i-1} \in A \cap x_i$. In neither case do we find an element disjoint with A. Hence, A violates Foundation. ■

N.B. To show that Foundation prohibits infinitely descending $\in$-chains, we'll need the Axiom of Infinity.

Some Standard Mathematical Definitions

$dom(R) =_{df} \{x \in \bigcup\bigcup R \mid \exists y\ (\langle x, y\rangle \in R)\}$

$ran(R) =_{df} \{y \in \bigcup\bigcup R \mid \exists x\ (\langle x, y\rangle \in R)\}$

Defn. $f : A \to B$ iff $Fn(f)\ \wedge\ dom(f) = A\ \wedge\ ran(f) \subseteq B$.

Defn. $f \restriction C =_{df} \{\langle x, y\rangle \in f \mid x \in C\}$

Defn. $f''C =_{df} \{f(x) \mid x \in C\}$

Defn. $f : A \to B$ is **injective** iff $\forall x_1, x_2 \in A(x_1 \neq x_2 \to f(x_1) \neq f(x_2))$.

Defn. $f : A \to B$ is **surjective** iff $\forall y \in B\ \exists x \in A(y = f(x))$, i.e., $ran(f) = B$.

Defn. $f : A \to B$ is a **bijection** iff f is both injective and surjective.

Well-Orderings

Let R and S be relations, and let A and B be sets. We do not assume that $dom(R) = A$ or $dom(S) = B$.

Defn. (Order Isomorphism) $\langle A, R\rangle \simeq \langle B, S\rangle$ iff there exists a bijection $f : A \to B$ s.t. $\forall x_1, x_2 \in A(\langle x_1, x_2\rangle \in R \leftrightarrow \langle f(x_1), f(x_2)\rangle \in S)$.

Defn. $\langle A, R\rangle$ is a **strict linear ordering** iff

1. $\forall x, y \in A\,(\langle x, y\rangle \in R \to \langle y, x\rangle \notin R)$
2. $\forall x, y, z \in A\,((\langle x, y\rangle \in R \,\wedge\, \langle y, z\rangle \in R) \to \langle x, z\rangle \in R)$
3. $\forall x, y \in A\,(\langle x, y\rangle \in R \,\vee\, \langle y, x\rangle \in R \,\vee\, x = y)$.

Defn. $\langle A, R\rangle$ is a **well-ordering** iff $\langle A, R\rangle$ is a strict linear ordering s.t. every non-empty subset of A has an R-least element, i.e.,

$$\forall B \subseteq A(B \neq 0 \to \exists x \in B\, \forall y \in B\, (\langle y, x\rangle \notin R)).$$

AC, Proper Initial Segments

This is enough for us to be able to state concisely an equivalent of the axiom of choice, i.e., equivalent given enough of the axioms of ZF:

(AC) Every set can be well-ordered, i.e.,

$$\forall A \exists R (\langle A, R \rangle \text{ is a well-ordering}).$$

Here we are only stating the principle for future reference, not adopting it as an axiom.

Proper Initial Segments

$$\mathit{pred}(A, x, R) =_{df} \{y \in A \mid \langle y, x \rangle \in R\}.$$

Lemma. If $\langle A, R \rangle$ is a well-ordering, then $\langle \mathit{pred}(A, x, R), R \rangle$ is a well-ordering.

Pf. Any sub-ordering of a well-ordering is a well-ordering. (Exercise)

Proper Initial Segments (cont.)

Lemma. No well-ordered set is isomorphic to any proper initial segment.

Pf. Let $\langle A, < \rangle$ be a well-ordering and $x \in A$. Suppose $f : \mathit{pred}(A, x, <) \to A$ is bijective. We show that f cannot be order-preserving. Suppose to the contrary that it is.

- Let $y = f^{-1}(x)$.
- Then $y \in \mathit{pred}(A, x, <)$ and since $x \notin \mathit{pred}(A, x, <)$, $y \neq x$.
- Hence, since $f(y) = f(f^{-1}(x)) = x$, $f(y) \neq y$.
- Thus $B =_{df} \{z \in \mathit{pred}(A, x, <) \mid f(z) \neq z\}$ is non-empty.
- Let m be the least element of B.
- Since $f(m) \neq m$, either (i) $f(m) < m$ or (ii) $m < f(m)$.

Proof of the Lemma (cont.)

We examine each case in turn.

Case (i) $f(m) < m$.

- Then $f(m) \in \mathit{pred}(A, x, <)$ but $f(m) \notin B$.
- Hence, $f(f(m)) = f(m)$.
- But, since f is 1-1, we apply f^{-1} to both sides to get $f(m) = m$, which contradicts the assumption $f(m) < m$.

Case (ii) $m < f(m)$

- By applying f^{-1} to both sides it follows that $f^{-1}(m) < m$.
- Hence $f^{-1}(m) \notin B$.
- Hence $f(f^{-1}(m)) = f(m)$.
- I.e., $m = f(m)$, contradicting $m < f(m)$. ■

Uniqueness of Isomorphisms

Lemma. Let $\langle A, <_R \rangle$ and $\langle B, <_S \rangle$ be isomorphic well-orderings. Then there exists a unique isomorphism $f : A \to B$.

Pf. Suppose that f and g are distinct isomorphisms. Let $C = \{x \in A \mid f(x) \neq g(x)\}$. Since C is non-empty, it has an R-least element y. Since $f(y) \neq g(y)$, we have either (i) $f(y) <_S g(y)$ or (ii) $g(y) <_S f(y)$.

Case (i). Suppose $f(y) <_S g(y)$.

1. Pull the two back by g^{-1}. Then $g^{-1}(f(y)) <_R y$.
2. Hence $f(g^{-1}(f(y))) <_S f(y)$.
3. But since y is the R-least element of C, it follows from (1) that $f(g^{-1}(f(y))) = g(g^{-1}(f(y))) = f(y)$, yielding a contradiction with (2).

Case (ii) is the same interchanging the roles of f and g.

Replacement Schema

- At this point we are [almost] poised to begin the development of the theory of ordinals as archetypes of well-ordered sets.
- One further result we want is that all well-orderings are comparable in that given any two, either they are isomorphic, or else one is isomorphic to a proper initial segment of the other. But to prove this, we'd like to use Cartesian products, but these are not yet guaranteed to exist. However, the following guarantees they exist.

Replacement Schema. Let $\varphi(x, z)$ be a wff s.t. $v \notin FV(\varphi(x, z))$. $[FV(\varphi(x, z))$ may contain variables in addition to x and z.] Then the universal closure of the following is an axiom.

$$\forall x \in A \, \exists! z \, \varphi(x, z) \to \exists v \, \forall x \in A \, \exists z \in v \, \varphi(x, z)$$

Replacement (cont.)

Note that A is playing the role of a variable in this formula, so that if $FV(\varphi(x,z)) = \{x,z\}$, then the universal closure of the corresponding instance of the Replacement Schema is:

$$\forall A(\forall x \in A\, \exists! z\, \varphi(x,z,) \to \exists v\, \forall x \in A\, \exists z \in v\, \varphi(x,z,)).$$

By Comprehension, then, for each set A that satisfies the antecedent condition, there is a unique set

$$V_A = \{z \mid \exists x \in A\, \varphi(x,z,w)\}.$$

So, in general, an instance of the Replacement Schema gives rise to an entire one parameter spectrum of additional sets, parameterized by those sets A satisfying the antecedent..

Replacement (cont.)

If $FV(\varphi(x,z)) = \{x, z, w\}$, then the corresponding instance of the Replacement Schema is:

$$\forall w \forall A (\forall x \in A \, \exists ! z \, \varphi(x,z,w) \to \exists v \, \forall x \in A \, \exists z \in v \, \varphi(x,z,w)),$$

which is equivalent to

$$\forall A \forall w (\forall x \in A \, \exists ! z \, \varphi(x,z,w) \to \exists v \, \forall x \in A \, \exists z \in v \, \varphi(x,z,w)).$$

For all A, w satisfying the antecedent, this gives rise (together with Comprehension) to a two parameter spectrum of sets

$$V(A,w) = \{z \mid \exists x \in A \, \varphi(x,z,w)\}.$$

We can take advantage of this to show that, for any sets A and B, there is a set corresponding to the collection of items

$$A \times B =_{df} \{\langle x, y \rangle \mid x \in A \ \wedge \ y \in B\}.$$

Existence of Cartesian Products

Let φ be the wff $z = \langle x, y \rangle$. Fix the sets A and B. If we pick any set y (in particular we have in mind an arbitrary element of B), then

$$\forall x \in A \, \exists! z \, (z = \langle x, y \rangle).$$

Thus by Replacement and Comprehension there is a unique set

$$prod(A, y) =_{df} \{z \mid \exists x \in A \, (z = \langle x, y \rangle)\}.$$

Since y was chosen arbitrarily, we have in particular

$$\forall y \in B \, \exists! u \, (u = prod(A, y)).$$

We now have, as another instance of Replacement, the universal generalization of:

$$\forall y \in B \exists! u \, (u = prod(A, y)) \to \exists v \, \forall y \in B \, \exists u \in v (u = prod(A, y)).$$

Existence of Cartesian Products (cont.)

Thus,

$$\exists v\, \forall y \in B\, \exists u \in v(u = \mathit{prod}(A, y)).$$

By Comprehension, an instance of v is the set

$$\begin{aligned} \mathit{prod}'(A, B) &= \{u \mid \exists y \in B\, (u = \mathit{prod}(A, y))\} \\ &= \{\mathit{prod}(A, y) \mid y \in B\}. \end{aligned}$$

Now $\mathit{prod}'(A, B)$ is a set of sets of ordered pairs, such that

$$\bigcup \mathit{prod}'(A, B) = \{\langle x, y\rangle \mid x \in A \ \wedge \ y \in B\},$$

i.e.,

$$A \times B =_{df} \bigcup \mathit{prod}'(A, B).$$

Comparison Theorem for Well-Ordered Sets

Theorem. Any two well-orderings are either isomorphic or else one is isomorphic to a proper initial segment of the other.

Pf. Let $\langle A, R\rangle$ and $\langle B, S\rangle$ be well-orderings. Let

$$f = \{\langle x, y\rangle \in A \times B \mid \langle pred(A, x, R), R\rangle \simeq \langle pred(B, y, S), S\rangle\}.$$

Claim 1: $Fn(f)$.

Sub-Pf. Suppose $\langle x, y_1\rangle \in f$ and $\langle x, y_2\rangle \in f$. Then there exist isomorphisms

$$g_1 : pred(A, x, R) \to pred(B, y_1, S)$$

and

$$g_2 : pred(A, x, R) \to pred(B, y_2, S).$$

Then

$$g_2 \circ g_1^{-1} : pred(B, y_1, S) \to pred(B, y_2, S)$$

is an isomorphism. If $y_1 \neq y_2$, then we have a well-ordering isomorphic to a proper initial segment of itself. Hence $y_1 = y_2$.

Comparison Theorem for Well-Ordered Sets (cont.)

Claim 2: f is injective.

Sub-Pf. This is equivalent to showing $Fn(f^{-1})$, which is done as before by reversing the roles of x and y.

Claim 3: $dom(f) = A$ or $ran(f) = B$ (or both).

Sub-Pf. Suppose $dom(f) \neq A$ and $ran(f) \neq B$. Let x_0 be the R-least element of $A \setminus dom(f)$ and y_0 the S-least element of $B \setminus ran(f)$. Then $\langle pred(A, x_0, R), R\rangle \simeq \langle pred(B, y_0, S), S\rangle$ (the isomorphism is f). So $\langle x_0, y_0\rangle \in f$.

Concluding argument.

- If $dom(f) = A$ and $ran(f) = B$, then f is an isomorphism from $\langle A, R\rangle$ to $\langle B, S\rangle$.
- If $dom(f) = A$ and $ran(f) \neq B$, then f is an isomorphism of $\langle A, R\rangle$ to some proper initial segment from $\langle B, S\rangle$.
- If $dom(f) \neq A$ and $ran(f) = B$, then f^{-1} is an isomorphism from $\langle B, S\rangle$ to a proper initial segment of $\langle A, R\rangle$. ■

The "Well-Ordering" of the Class of Equivalence Classes of Well-Ordered Sets

Suppose that $\langle A, R\rangle$ and $\langle B, S\rangle$ are well-orderings.

Notation. $[\langle A, R\rangle] =_{df} \{\langle B, S\rangle \mid \langle A, R\rangle \simeq \langle B, S\rangle\}$.

Notation. $[\langle A, R\rangle] \blacktriangleleft [\langle B, S\rangle]$ iff $\langle A, R\rangle$ is isomorphic to a proper initial segment of $\langle B, S\rangle$.

Observation. $\blacktriangleleft$ has the effect of "well-ordering" the class of all equivalence classes of well-ordered sets.

Exercise. Prove it!

Desideratum. Want to select out of each equivalence class a unique representative for the class, indeed in such a way that $\in$ has the effect of well-ordering these representatives. The class of these representatives will be the *ordinals*.