Ordinals and Cardinals

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Introducing Ordinals

Defn. A set is **transitive** iff every element is a subset, i.e., $\mathit{trans}(x)$ iff $\forall y(y \in x \to y \subseteq x)$.

N.B. x is transitive iff if $z \in y$ and $y \in x$, then $z \in x$.

Defn. For any set A, $\epsilon_A =_{df} \{\langle x, y\rangle \in A \times A \mid x \in y\}$. As a notational convenience, write $x \in_A y$ for $\langle x, y\rangle \in \epsilon_A$.

Defn. $ON(x)$ (i.e., x is an **ordinal**) iff $\mathit{trans}(x)$ and ϵ_x well-orders x.

Notation. Let ZF^- be ZF w/o Foundation. We have yet to add the Axiom of Infinity (Inf) and the the Power Set Axiom (P). So the system we're currently working in is ZF^-- Inf - P.

Convention. We typically use lower case letters from the beginning of the Greek alphabet, i.e., $\alpha, \beta, \gamma, \ldots$ to range over ordinals.

Elements of Ordinals Are Ordinals

Proposition. ZF$^-$- Inf - P $\vdash \forall\alpha\forall x(ON(\alpha) \wedge x \in \alpha \to ON(x))$.

Pf. Suppose $ON(\alpha)$ and $x \in \alpha$. In order to show $ON(x)$ we need to show two things,

1. ϵ_x well-orders x, and
2. $trans(x)$.

(1) The first of these is the easier. Since $trans(\alpha)$, it follows $x \subseteq \alpha$. So,

$$\begin{aligned} ON(\alpha) &\Rightarrow \epsilon_\alpha \text{ well-orders } \alpha \\ &\Rightarrow \epsilon_\alpha \text{ well-orders } x \\ &\Rightarrow \epsilon_x \text{ well-orders } x. \end{aligned}$$

Elements of Ordinals Are Ordinals (cont.)

(2) To show that *trans*(x), we need to show that any $y \in x$ is s.t. $y \subseteq x$, in other words, for any $z \in y$, also $z \in x$. So suppose $y \in x$ and $z \in y$.

- *trans*(α) and $x \in \alpha \Rightarrow x \subseteq \alpha$.
- $x \subseteq \alpha$ and $y \in x \Rightarrow y \in \alpha$.
- *trans*(α) and $y \in \alpha \Rightarrow y \subseteq \alpha$, and hence $z \in \alpha$.
- $x, y, z \in \alpha$ and $z \in y$ and $y \in x \Rightarrow z \in_\alpha y$ and $y \in_\alpha x$.
- $z \in_\alpha y$ and $y \in_\alpha x \Rightarrow z \in_\alpha x$ (because $\in_\alpha$ well-orders α).
- $z \in_\alpha x \Rightarrow z \in x$.

■

Any Element of an Ordinal is the Proper Initial Segment it Cuts Off

To compactify notation, let T temporarily stand for ZF$^-$- Inf - P.

Proposition. $\mathrm{T} \vdash \forall\alpha\forall x(ON(\alpha) \wedge x \in \alpha \rightarrow x = pred(\alpha, x, \epsilon_\alpha))$.

Pf. Suppose $ON(\alpha)$ and $x \in \alpha$.

(1) Suppose $y \in x$. Since $trans(\alpha)$, $x \subseteq \alpha$, and hence $y \in \alpha$. Thus, $y \in_\alpha x$ and hence $y \in pred(\alpha, x, \epsilon_\alpha)$. Therefore $x \subseteq pred(\alpha, x, \epsilon_\alpha)$.

(2) Conversely, suppose $y \in pred(\alpha, x, \epsilon_\alpha)$. Then $y \in_\alpha x$. Hence $y \in x$. Therefore, $pred(\alpha, x, \epsilon_\alpha) \subseteq x$.

From (1) and (2) we conclude $x = pred(\alpha, x, \epsilon_\alpha)$. ■

Isomorphic Ordinals Are Identical

Proposition. $\mathsf{T} \vdash \forall\alpha\forall\beta(ON(\alpha) \wedge ON(\beta) \wedge \langle\alpha, \epsilon_\alpha\rangle \simeq \langle\beta, \epsilon_\beta\rangle \to \alpha = \beta)$.

Pf. Suppose $\langle\alpha, \epsilon_\alpha\rangle \simeq \langle\beta, \epsilon_\beta\rangle$. Let f be the unique isomorphism from $\langle\alpha, \epsilon_\alpha\rangle$ to $\langle\beta, \epsilon_\beta\rangle$. Suppose f is not the identity function on α. Let $\gamma \in \alpha$ be the ϵ_α-least element s.t. $f(\gamma) \neq \gamma$. Then for any δ,

$$\begin{aligned}
\delta \in \gamma \quad &\Leftrightarrow \quad \delta \in_\alpha \gamma \\
&\Leftrightarrow \quad f(\delta) \in_\beta f(\gamma) \quad [f \text{ being an isomorphism}] \\
&\Leftrightarrow \quad \delta \in_\beta f(\gamma) \quad [\text{since } f(\delta) = \delta \text{ if } \delta \in_\alpha \gamma\ ] \\
&\Leftrightarrow \quad \delta \in f(\gamma).
\end{aligned}$$

By extensionality, $f(\gamma) = \gamma$, yielding a contradiction. So, f is the identity function on α, and thus $\alpha = \beta$.

Given Distinct Ordinals, One is a Member of the Other

Proposition. $\mathsf{T} \vdash \forall\alpha\forall\beta(\mathit{ON}(\alpha) \wedge \mathit{ON}(\beta) \rightarrow \alpha \in \beta \mathbin{\dot\vee} \alpha = \beta \mathbin{\dot\vee} \beta \in \alpha)$.

Pf Suppose $\mathit{ON}(\alpha)$ and $\mathit{ON}(\beta)$. Then, either $\langle\alpha, \epsilon_\alpha\rangle \simeq \langle\beta, \epsilon_\beta\rangle$ xor one is isomorphic to a proper initial segment of the other. In the former case, $\alpha = \beta$. So consider the latter, say there exists a $\gamma \in \beta$ such that

$$\langle\alpha, \epsilon_\alpha\rangle \simeq \langle \mathit{pred}(\beta, \gamma, \epsilon_\beta), \epsilon_\beta\rangle.$$

Then $\alpha = \gamma$, and therefore $\alpha \in \beta$. On the other hand, if there exists a $\gamma \in \alpha$ such that

$$\langle\beta, \epsilon_\beta\rangle \simeq \langle \mathit{pred}(\alpha, \gamma, \epsilon_\alpha), \epsilon_\alpha\rangle,$$

then $\beta = \gamma$, and therefore $\beta \in \alpha$. ■

$\in$ is Transitive on the Class of Ordinals

Proposition.
$\mathsf{T} \vdash \forall\alpha,\beta,\gamma((\mathit{ON}(\alpha) \wedge \mathit{ON}(\beta) \wedge \mathit{ON}(\gamma) \wedge \alpha \in \beta \ \wedge \ \beta \in \gamma) \rightarrow \alpha \in \gamma)$.

Pf. Let α, β, γ be ordinals and suppose $\alpha \in \beta$ and $\beta \in \gamma$. Since *trans*(γ), it follows that $\beta \subseteq \gamma$, and hence $\alpha \in \gamma$. ■

Scholium. The entity $\in$ is not a relation on the class of ordinals. Rather it is a class of relational type.

Any Non-Empty Set C of Ordinals Has an ϵ_C-Least Element

Proposition. The following is provable in ZF$^-$−Inf−P:

$$\forall C((C \neq \emptyset \ \wedge \ \forall x(x \in C \rightarrow ON(x))) \rightarrow \exists x \in C \ \forall y(y \in C \rightarrow y \notin_C x)).$$

Pf. Let C be a non-empty set of ordinals. Pick any $z \in C$.

(1) Suppose $z \cap C = \emptyset$. Let $y \in C$. Then $y \notin z$, else $z \cap C \neq \emptyset$. Hence, $y \notin_C z$ and therefore z is the $\in_C$-least element.

(2) Alternatively, suppose $z \cap C \neq \emptyset$. Since $z \cap C \subseteq z$, $z \cap C$ has an ϵ_z-least element, call it m. Now let y be an arbitrary element of C. If $y \in m$, then $y \in z$, and hence $y \in z \cap C$ and m is not the ϵ_z-least element of $z \cap C$. Therefore, either $m \in y$ or $y = m$, i.e., $m \in_C y$ unless $y = m$ ■

Burali-Forti Paradox

Let ON be the set of all ordinals.

Since all the members of an ordinal are ordinals, $\alpha \subseteq \mathit{ON}$ for any $\alpha \in \mathit{ON}$. So, ON is transitive.

Furthermore, $\in_{\mathit{ON}}$ well orders ON since it is a strict linear ordering on ON and every non-empty set of ordinals has an $\in$-least element.

Therefore ON itself is an ordinal!

Hence $\mathit{ON} \in \mathit{ON}$.

But it was shown (w/o appeal to Foundation) that no ordinal is an element of itself.

Resolution of the Burali-Forti Paradox

Proposition. $\mathsf{ZF}^- - \mathsf{Inf} - \mathsf{P} \vdash \neg\exists x \forall y (ON(y) \to y \in x)$.
In words: it is provable from ZF$^-$−Inf−P that there is no set containing all ordinals.

Pf. Suppose otherwise, i.e., that such an x exists. Then, by Comprehension,

$$ON =_{df} \{y \in x \mid ON(y)\}$$

is the set of all ordinals. ON is transitive since every element of an ordinal is an ordinal. Furthermore, ϵ_{ON} well-orders ON, since

1. $\forall \alpha, \beta \in ON(\alpha \in_{ON} \beta \ \dot{\vee} \ \alpha = \beta \ \dot{\vee} \ \beta \in_{ON} \alpha)$, where $\dot{\vee}$ is exclusive disjunction (XOR). This entails both asymmetry and trichotomy.
2. $\forall \alpha, \beta, \gamma \in ON((\alpha \in_{ON} \beta \ \wedge \ \beta \in_{ON} \gamma) \to \alpha \in_{ON} \gamma)$, i.e., $\in_{ON}$ is a transitive relation.
3. Every non-empty subset of ON has an ϵ_{ON}-least element.

Hence $ON(ON)$, i.e., $ON \in ON$, and hence $ON \in_{ON} ON$. But $ON = ON$, excluding $ON \in_{ON} ON$ by (1) above. ■

Any Transitive Set of Ordinals Is an Ordinal

Abusive Notation. Write $x \in ON$ for $ON(x)$ and

$$ON = \{x \mid ON(x)\}$$

for the extension of $ON(x)$.

Lemma. $\mathrm{ZF}^{-} - \mathrm{Inf} - \mathrm{P} \vdash$ Any transitive set of ordinals is an ordinal.

Pf. Suppose that A is a transitive set of ordinals. Since we're already given that A is transitive, it suffices to show that ϵ_A well-orders A. But this follows from the analogues of (1)–(3) in the proof of the Burali-Forti paradox. ■

A Uniqueness of Representation Result

Lemma (ZF$^-$−Inf−P). Let $\langle A, R\rangle$ be a well-ordering. If $\exists\alpha \in ON$ such that $\langle A, R\rangle \simeq \langle \alpha, \epsilon_\alpha\rangle$, then $\exists!\alpha \in ON$ such that $\langle A, R\rangle \simeq \langle \alpha, \epsilon_\alpha\rangle$.

Pf. Suppose $\langle A, R\rangle \simeq \langle \alpha, \epsilon_\alpha\rangle$ and $\langle A, R\rangle \simeq \langle \beta, \epsilon_\beta\rangle$. Then $\langle \alpha, \epsilon_\alpha\rangle \simeq \langle \beta, \epsilon_\beta\rangle$, and thus $\alpha = \beta$. ■

Representation Theorem for Well-Ordered Sets

Theorem ($\mathsf{ZF}^- - \mathsf{Inf} - \mathsf{P}$). Any well-ordering is isomorphic to a unique ordinal.

Pf. Suppose $\langle A, R\rangle$ is a well-ordering. Let

$$B = \{x \in A \mid \exists \alpha \in ON(\langle pred(A, x, R), R\rangle \simeq \langle \alpha, \epsilon_\alpha\rangle)\}.$$

By the last lemma

$$\forall x \in B\ \exists! \alpha \in ON(\langle pred(A, x, R), R\rangle \simeq \langle \alpha, \epsilon_\alpha\rangle).$$

Hence, by Replacement,

$$\exists Y \subseteq ON\ \forall x \in B\ \exists \alpha \in Y(\langle pred(A, x, R), R\rangle \simeq \langle \alpha, \epsilon_\alpha\rangle).$$

Further by Comprehension, there is a set

$$C = \{\alpha \in Y \mid \exists x \in B(\langle pred(A, x, R), R\rangle \simeq \langle \alpha, \epsilon_\alpha\rangle)\}.$$

Representation Theorem for Well-Ordered Sets (cont.)

Now let

$$f = \{\langle x, \alpha \rangle \in B \times C \mid (\langle \mathit{pred}(A, x, R), R \rangle \simeq \langle \alpha, \epsilon_\alpha \rangle)\}.$$

Claim. $f : B \to C$ is bijective:
By construction $\mathit{Fn}(f)$ with $\mathit{dom}(f) = B$ and $\mathit{ran}(f) = C$. Further, f is 1-1, else for some $x \in B$, $\langle \mathit{pred}(B, x, R), R \rangle$ is isomorphic to some proper initial segment of itself.

Claim. $C \in \mathit{ON}$:
It suffices to show that $\mathit{trans}(C)$. So, pick $\alpha \in C$. Let $x = f^{-1}(\alpha)$, and let $\beta \in \alpha$. Since $\langle \mathit{pred}(A, x, R), R \rangle \simeq \langle \alpha, \epsilon_\alpha \rangle$, there exists an isomorphism $g : \mathit{pred}(A, x, R) \to \alpha$. Let $y = g^{-1}(\beta)$. Clearly g restricted to $\mathit{pred}(A, y, R)$ is an isomorphism from $\langle \mathit{pred}(A, y, R), R \rangle$ to $\langle \beta, \epsilon_\beta \rangle$. So $\beta \in C$.

Representation Theorem for Well-Ordered Sets (cont.)

Claim. $A = B$.

Suppose $A \setminus B \neq \emptyset$. Let m be the R-least element of $A \setminus B$. Then $B = \mathit{pred}(A, m, R)$. But we've just shown that f is an isomorphism from $\langle B, R \rangle$ to $\langle C, \epsilon_C \rangle$ and that $C \in ON$. Hence $m \in B$, yielding a contradiction. ■

Notation and Conventions

Notation. Suppose $\langle A, R\rangle$ a well-ordering. Write

$$\mathit{type}(A, R)$$

for the unique ordinal α s.t. $\langle A, R\rangle \simeq \langle \alpha, \epsilon_\alpha\rangle$.

Convention. From now on, suppress $ON(\cdot)$ when quantifying over ordinals $\alpha, \beta, \gamma, \ldots$, i.e., we relativize to ON, i.e., for any wff φ:

- $\forall\alpha\ \varphi$ is short for $\forall\alpha(ON(\alpha) \to \varphi)$, and
- $\exists\alpha\ \varphi$ is short for $\exists\alpha(ON(\alpha) \wedge \varphi)$.

Notation. For ordinals α and β, write:

$$\begin{aligned} \alpha < \beta &\quad \text{for} \quad \alpha \in \beta \\ \alpha \leq \beta &\quad \text{for} \quad \alpha < \beta \text{ or } \alpha = \beta \end{aligned}$$

Some Lemmata and Definitions

Lemma. $\mathsf{ZF}^{-}-\mathsf{Inf}-\mathsf{P} \vdash \forall\alpha\forall\beta(\alpha \leq \beta \leftrightarrow \alpha \subseteq \beta)$

Pf. Exercise. [Keep in mind this is true only for *ordinals*.]

Defns. Let X be a set of ordinals.

$$\begin{aligned} \sup(X) &=_{df} \bigcup X \\ \min(X) &=_{df} \bigcap X \text{ if } X \neq \emptyset \end{aligned}$$

Lemma. The terminology is justified, i.e.

- $\sup(X)$ is the least ordinal α s.t. $\beta \leq \alpha$ for each $\beta \in X$, and
- $\min(X)$ is the least ordinal in X.

That is, sup and min are least upper bound and greatest lower bound, respectively.

Pf. Exercise.

Successor and Limit Ordinals, Natural Numbers

Defn. For any set x, $S(x) =_{df} x \cup \{x\}$.

Lemma.

1. $\forall\alpha\ ON(S(\alpha))$
2. $\forall\alpha\ \ \alpha < S(\alpha)$
3. $\forall\alpha\forall\beta(\beta < S(\alpha) \leftrightarrow \beta \leq \alpha)$

Pf. Exercise.

Defn. α is a **successor ordinal** $[SuccON(\alpha)]$ iff $\exists\beta\ \ \alpha = S(\beta)$.

Defn. α is a **limit ordinal** iff $\alpha \neq 0$ and α is not a successor ordinal.

Defn. α is a **natural number** $[\mathbb{N}(\alpha)]$ iff $\forall\beta \leq \alpha(\beta = 0\ \vee\ SuccON(\beta))$, i.e., for all $\beta \leq \alpha$, β is not a limit ordinal.

“Peano Postulates”

Theorem (ZF$^-$−Inf−P).

1. $\mathbb{N}(0)$
2. $\forall x(\mathbb{N}(x) \to \mathbb{N}(S(x)))$
3. $\forall x: \ 0 \neq S(x)$
4. $\forall x \forall y(S(x) = S(y) \to x = y)$
5. *Induction Schema*: Let $\varphi(x)$ be a wff s.t. $x \in FV(\varphi)$. Then the universal closure of the following holds.

$$[\varphi(0) \land \forall x \in \mathbb{N}\, (\varphi(x) \to \varphi(S(x)))] \to \forall x \in \mathbb{N}\; \varphi(x).$$

Comment on Notation. Since at this point, the natural numbers are given only as a class and not necessarily as a set, the notation $x \in \mathbb{N}$ is just a variant for predication: $\mathbb{N}(x)$. Written out in full, the displayed wff is:

$$[\varphi(0) \land \forall x(\mathbb{N}(x) \to (\varphi(x) \to \varphi(S(x))))] \to \forall x(\mathbb{N}(x) \to \varphi(x)).$$

Scholium on Classes

Technically speaking, a class is simply a predicate, i.e., a wff C in the language of ZF. It is useful, though, to think of the class as the extension of the predicate. E.g., suppose $FV(C) = \{x\}$. Then

$$\mathbf{C} = \{x \mid C(x)\}$$

can be thought of as a way of expressing this extension: "**C** is the collection/class of x's that meet the condition imposed by $C(x)$."

Some standard notation:

- $\mathbf{V} = \{x \mid x = x\}$
- $\mathbf{ON} = \{x \mid ON(x)\}$
- $\mathbf{WF} = \{x \mid (x \neq \emptyset \to \exists y \in x \; y \cap x = \emptyset)\}$
- $\in = \{\langle x, y\rangle \mid x \in y\}$

Some Class Notation

Notational Conventions.

- Use boldface letters to indicate proper classes rather than sets.
- $x \in \mathbf{C}$ means just $C(x)$.
- $\mathbf{C_1} \subseteq \mathbf{C_2}$ means $\forall x(C_1(x) \rightarrow C_2(x))$.
- $\mathbf{C_1} \cap \mathbf{C_2} = \{x \mid C_1(x) \wedge C_2(x)\}$.
  (So that $x \in \mathbf{C_1} \cap \mathbf{C_2}$ means just $C_1(x) \wedge C_2(x)$.)
- $x \cap \mathbf{C}$ means $\{y \in x \mid C(y)\}$.
  N.B. This is a set by Comprehension.

Proof of the "Peano Postulates"

1. $\mathbb{N}(0)$
2. $\forall x(\mathbb{N}(x) \to \mathbb{N}(S(x)))$
3. $\forall x :\ 0 \neq S(x)$
4. $\forall x \forall y (S(x) = S(y) \to x = y)$

Pf. (1)–(4) are trivial to prove. Only the induction schema:

$$[\varphi(0) \wedge \forall x \in \mathbb{N}\, (\varphi(x) \to \varphi(S(x)))] \to \forall x \in \mathbb{N}\ \varphi(x).$$

requires finesse. It helps to have the following.

Lemma Schema ($\mathsf{ZF}^{-}{-}\mathsf{Inf}{-}\mathsf{P}$). If $\mathbf{C} \subseteq \mathbb{N}$ and $\mathbf{C} \neq \emptyset$, then $\mathbf{C}$ has a least element.

Pf. Pick $z \in \mathbf{C}$. If $z \cap \mathbf{C} = \emptyset$, then z is the least element. Else the least element of $z \cap \mathbf{C}$ is the least element. ■

Proof of the “Peano Postulates” (cont.)

Proof of the induction schema then proceeds as follows:

Suppose $\varphi(0)$ and $\forall x \in \mathbb{N}\,(\varphi(x) \to \varphi(S(x)))$. We want to show that $\forall x \in \mathbb{N}\,\varphi(x)$. So suppose $\exists x \in \mathbb{N}\,\neg\varphi(x)$. Let

$$\mathbf{C} = \{x \in \mathbb{N} \mid \neg\varphi(x)\}.$$

Since $\mathbf{C} \neq \emptyset$, $\mathbf{C}$ has a least element, call it m. We have that $m \neq 0$ on pain of contradiction with $\varphi(0)$. So m is a successor ordinal. Let m' be its predecessor. Since m is the least element of $\mathbf{C}$, we have $\varphi(m')$. But then, by supposition and modus ponens, we have that $\varphi(m)$, contradicting the assumption that $m \in \mathbf{C}$. ■

Transfinite Induction on **ON**

Theorem Schema ($\mathrm{ZF}^{-}-\mathrm{Inf}-\mathrm{P}$). If $\mathbf{C} \subseteq \mathbf{ON}$ and $\mathbf{C} \neq \emptyset$, then $\mathbf{C}$ has a least element.

Pf. Same as for $\mathbb{N}$: Pick $z \in \mathbf{C}$. If $z \cap \mathbf{C} = \emptyset$, then z is the least element. Else the least element of $z \cap \mathbf{C}$ is the least element. ■

Transfinite Induction Schema for Ordinals ($\mathrm{ZF}^{-}-\mathrm{Inf}-\mathrm{P}$). If

1. $\varphi(0)$,
2. $\forall\alpha(\varphi(\alpha) \to \varphi(S(\alpha)))$, and
3. for each limit ordinal γ, if $\varphi(\xi)$ for all $\xi < \gamma$, then $\varphi(\gamma)$;

then $\forall\alpha\ \varphi(\alpha)$.

Pf. Suppose (1)–(3) but $\exists\alpha\neg\varphi(\alpha)$. Let $\mathbf{C} = \{\alpha \mid \neg\varphi(\alpha)\}$. Let μ be the least element of C. μ must be either 0, a successor ordinal, or a limit ordinal. But each case is ruled out by (1)–(3), respectively. ■

Axiom of Infinity

Axiom of Infinity. $\exists x(0 \in x \wedge \forall y(y \in x \rightarrow S(y) \in x))$.

Lemma (ZF$^-$−P). Suppose A satisfies the axiom of infinity. If $n \in \mathbb{N}$, then $n \in A$.

Pf. Do induction on $\mathbb{N}$ using the wff $n \in A$ for φ.

Lemma. $\exists! x \forall y(y \in x \leftrightarrow \mathbb{N}(y))$.

Pf. From the axiom of infinity by comprehension and extensionality.

Defn. $\omega = \{x \mid \mathbb{N}(x)\}$.

Lemma. ω is the smallest limit ordinal.

Pf. Exercise.

Ordinal Addition

Defn. $\alpha + \beta = \mathit{type}(\alpha \times \{0\} \cup \beta \times \{1\}, R)$, where

$$\begin{aligned} R \quad = \quad & \{\langle\langle\gamma, 0\rangle, \langle\delta, 0\rangle\rangle \mid \gamma < \delta < \alpha\} \\ & \cup \{\langle\langle\gamma, 1\rangle, \langle\delta, 1\rangle\rangle \mid \gamma < \delta < \beta\} \\ & \cup \{\langle\langle\gamma, 0\rangle, \langle\delta, 1\rangle\rangle \mid \gamma < \alpha \wedge \delta < \beta\}. \end{aligned}$$

Theorem ($\mathrm{ZF}^{-}-\mathrm{Inf}-\mathrm{P}$). For all α, β, γ:

1. $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$
2. $\alpha + 0 = \alpha$
3. $\alpha + 1 = S(\alpha)$
4. $\alpha + S(\beta) = S(\alpha + \beta)$
5. If β is a limit ordinal, then $\alpha + \beta = \sup\{\alpha + \xi \mid \xi < \beta\}$.

Pf. Exercise.

N.B. Frequently, (2),(4), and (5) are used to give a definition of ordinal addition by *transfinite recursion*, but we have yet to justify that method of definition.

Remarks on Commutativity

Corollary. $\mathsf{ZF}^{-}-\mathsf{Inf}-\mathsf{P} \vdash \forall \alpha, \beta \in \mathbb{N}\ (\alpha + \beta = \beta + \alpha)$.

Pf. First use the obvious instance of the induction scheme for Peano Arithmetic (PA) to show that $\forall \alpha \in \mathbb{N}\ (\alpha + 0 = 0 + \alpha)$. Then let φ be be $\alpha + \beta = \beta + \alpha$, and use the following instance of the induction schema for PA:

$$\forall \alpha([\varphi(\alpha, 0) \wedge \forall \beta(\varphi(\alpha, \beta) \rightarrow \varphi(\alpha, S(\beta)))] \rightarrow \forall \beta \varphi(\alpha, \beta)).$$

■

Proposition ($\mathsf{ZF}^{-}-\mathsf{P}$). Suppose $\alpha \neq 0$. Then $\omega < \omega + \alpha$. But if $\alpha < \omega$, then $\alpha + \omega = \omega$.

Pf. Exercise.

The Lexicographic Ordering

Defn. Let $\langle A, R\rangle$ and $\langle B, S\rangle$ be strict orderings of any sort, perhaps even non-linear. The *lexicographic* (or dictionary) order L on $A \times B$ is

$$L =_{df} \{\langle\langle x_1, y_1\rangle, \langle x_2, y_2\rangle\rangle \mid \langle x_1, x_2\rangle \in R \ \vee \ (x_1 = x_2 \ \wedge \langle y_1, y_2\rangle \in S)\}.$$

Lemma. If $\langle A, R\rangle$ and $\langle B, S\rangle$ are well-orderings and L is the lexicographic order on $A \times B$, then $\langle A \times B, L\rangle$ is a well-ordering.

Pf. Exercise.

Ordinal Multiplication

Defn. $\alpha \cdot \beta = \mathit{type}(\beta \times \alpha, L)$, where L is the lexicographic order on $\beta \times \alpha$.

Theorem (ZF$^-$−Inf−P). For all α, β, γ:

1. $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$
2. $\alpha \cdot 0 = 0$
3. $\alpha \cdot 1 = \alpha$
4. $\alpha \cdot S(\beta) = \alpha \cdot \beta + \alpha$
5. If β is a limit ordinal, then $\alpha \cdot \beta = \sup\{\alpha \cdot \xi \mid \xi < \beta\}$
6. $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.

Pf. Exercise.

N.B. Sometimes ordinal multiplication is defined recursively as iterated addition via clauses (2), (4) and (5). But again, we have not yet justified transfinite recursion.

Remarks on Commutativity and Distributivity

Proposition. $\mathsf{ZF}^- - \mathsf{Inf} - \mathsf{P} \vdash \forall \alpha, \beta \in \mathbb{N}\, (\alpha \cdot \beta = \beta \cdot \alpha)$.

Pf. By the usual inductive methods of Peano Arithmetic.

Proposition ($\mathsf{ZF}^- - \mathsf{P}$). Suppose $1 < \alpha$. Then $\omega < \omega \cdot \alpha$, but if $\alpha < \omega$, then $\alpha \cdot \omega = \omega$.

Pf. Exercise.

Proposition ($\mathsf{ZF}^- - \mathsf{Inf} - \mathsf{P}$). $\forall \alpha, \beta, \gamma \in \mathbb{N}\, ((\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma)$.

Pf. Exercise.

Proposition ($\mathsf{ZF}^- - \mathsf{P}$). Suppose $0 < \alpha, \beta$. Then $\omega < \alpha \cdot \omega + \beta \cdot \omega$, but if $\alpha, \beta < \omega$, then $(\alpha + \beta) \cdot \omega = \omega$.

Pf. Exercise.

Interlude: Sequences

Let A be a non-empty set.

Defn. A *finite sequence* s on A is a function $s : n \to A$ for some $n \in \mathbb{N}$. We typically write $s_0, s_1, \ldots s_{n-1}$ for $s(0), s(1), \ldots, s(n-1)$.

Defn. A sequence of length $\alpha \in \mathbf{ON}$ is a function $s : \alpha \to A$. Again, typically write s_ξ for $s(\xi)$ for all $\xi < \alpha$.

Defn. For an arbitrary set I, a sequence on A indexed by I is a function $s : I \to A$. For $i \in I$, write s_i for $s(i)$.

Defn. Let $s : \alpha \to A$ and $t : \beta \to B$ be ordinal length sequences. The *concatenation* $s^\frown t$ of s with t is the function

$$s^\frown t : \alpha + \beta \to A \cup B$$

s.t. $(s^\frown t) \restriction \alpha = s$ and $(s^\frown t)(\alpha + \xi) = t(\xi)$ for all $\xi < \beta$.

Interlude: Sets of Sequences

Defn. If I is an arbitrary index set, let

$$\mathbf{Maps}(I, A) =_{df} \{s \mid \ s : I \to A\}.$$

Without the power set axiom, it cannot be shown in general that $\mathbf{Maps}(I, A)$ is a set. In particular, $\mathbf{Maps}(\omega, 2)$ is the class of characteristic functions of subsets of ω.

Nonetheless, we can show in $\mathrm{ZF}^- - \mathrm{Inf} - \mathrm{P}$ that $\mathbf{Maps}(n, A)$ is a set for each $n \in \mathbb{N}$, and furthermore show in $\mathrm{ZF}^- - \mathrm{P}$ that the class of all finite sequences on A is a set.

Notation. Write ${}^{B}A$ for $\mathbf{Maps}(B, A)$ if $\mathbf{Maps}(B, A)$ is a set.

Sets of Sequences (cont.)

E.g., we can show in $\mathrm{ZF}^- - \mathrm{Inf} - \mathrm{P}$ that ${}^2A$ exists.

- First introduce the projection function symbols π_0 and π_1 s.t. for any ordered pair $x = \langle y, z\rangle$, $\pi_0 x = y$ and $\pi_1 x = z$.
- Let $\varphi = \varphi(x, f)$ in the Replacement Scheme be the wff

$$(f{:}\, 2 \to A \ \wedge\ f(0) = \pi_0 x \ \wedge\ f(1) = \pi_1 x).$$

- Clearly we can show in $\mathrm{ZF}^- - \mathrm{Inf} - \mathrm{P}$ that

$$\forall x \in A \times A\ \exists! f (f{:}\, 2 \to A \ \wedge\ f(0) = \pi_0 x \ \wedge\ f(1) = \pi_1 x).$$

- Then by Replacement

$$\exists Z\ \forall x \in A \times A\ \exists f \in Z\ \varphi(x, f).$$

- Let Y be such a Z. Then by Comprehension and Extensionality

$${}^2A =_{df} \{f \in Y \mid \exists x \in A \times A\ \ \varphi(x, f)\}.$$

Sets of Sequences (cont.)

Lemma (ZF$^-$$-Inf-$P). ${}^nA$ exists for each $n \in \mathbb{N}$.

Pf. The argument proceeds by induction:
For $n = 0$, we have ${}^0A = 0$.
For the inductive step, suppose ${}^nA$ exists. We show that ${}^{(n+1)}A$ exists as follows.

- For any $x \in {}^nA \times A$ there exists a unique $f : n+1 \to A$ s.t. $f \upharpoonright n = \pi_0 x$ and $f(n) = \pi_1 x$.
- Hence, by Replacement there exists a Z s.t. for any $x \in {}^nA \times A$ there exists an $f \in Z$ s.t. $f : n+1 \to A$ and $f \upharpoonright n = \pi_0 x$ and $f(n) = \pi_1 x$.
- By Comprehension and Extensionality we can define ${}^{(n+1)}A$ to be

$$\{f \in Z \mid \exists x \in {}^nA \times A\, ((f : n+1 \to A) \wedge f \upharpoonright n = \pi_0 x \wedge f(n) = \pi_1 x)\}.$$

■

The Set of All Finite Sequences on a Set

Lemma ($\mathsf{ZF}^{-}{-}\mathsf{P}$). For any set A, $\{{}^{n}A \mid n \in \omega\}$ is a set.

Pf. Since each ${}^{n}A$ exists and is unique,

$$\forall n \in \omega\, \exists ! X \forall f (f \in X \leftrightarrow f : n \to A)$$

is provable in $\mathsf{ZF}^{-}{-}\mathsf{P}$. Hence, by Replacement, there exists a Z s.t. ${}^{n}A \in Z$ for each $n \in \omega$. By Comprehension and Extensionality, there exists a unique $Y = \{{}^{n}A \mid n \in \omega\}$.

Defn. The set of all finite sequences on A is

$${}^{<\omega}A =_{df} \bigcup \{{}^{n}A \mid n \in \omega\}.$$

Scholium. It's standard to define the set A^n of n-tuples on A by recursion: $A^1 = A$, $A^{n+1} = A^n \times A$. Then $A^{<\omega}$ is the set of all tuples. These are all distinct from their l.h. cousins, though standing in a clear 1-1 relation.

Ordinal Exponentiation

It's possible, though complex, to give a purely combinatorial definition of ordinal exponentiation. (See Kunen, Exercise 7, Chapt. 1.) Far more intuitive is to define it by recursion as iterated multiplication.

Defn. α^{β} is define by transfinite recursion on β as follows.

- $\alpha^{0} = 1$.
- $\alpha^{S(\beta)} = \alpha^{\beta} \cdot \alpha$.
- If β is a limit ordinal, then $\alpha^{\beta} = \sup\{\alpha^{\xi} \mid \xi < \beta\}$.

Now that the pattern of transfinite recursion is clear, we can inquire into its justification.

Further Scholium on Classes

Recall that, technically speaking, a class is just a predicate, i.e., a wff C, with one *or more* free variables. This raises some puzzles when we then use the notation $\mathbf{C}$.

(1) For example, in the case of one free variable we are invited to write:

$$\mathbf{C} = \{x \mid C(x)\}.$$

For suppose x and y are distinct variables. Then $C(x)$ and $C(y)$ are distinct wff's, yet

$$\{x \mid C(x)\} = \{y \mid C(y)\}.$$

Do we have two classes or one? Note that the effect of the "braces and stroke" is to bind the variable occurring therein.

Further Scholium on Classes (cont.)

(2) For another, in the case of more than one variable, say $x \in y$, we are invited to write

$$\in = \{\langle x, y\rangle \mid x \in y\}$$

and to talk in terms of classes of ordered pairs. But as understood in the object language or in the metalanguage?

Further Scholium on Classes (cont.)

(3) One example of a class that Kunen gives (p. 24) is that of the "union operation"

$$\mathbf{UN} = \{\langle\langle x, y\rangle, z\rangle \mid z = x \cup y\}.$$

He says

> *Intuitively,* $\mathbf{UN} : \mathbf{V} \times \mathbf{V} \to \mathbf{V}$.

But what does *that* mean?

As a start, you might say

$$\begin{aligned} \mathbf{V} \times \mathbf{V} &= \{\langle x, y\rangle \mid x \in \mathbf{V} \wedge y \in \mathbf{V}\} \\ &= \{\langle x, y\rangle \mid x = x \ \wedge \ y = y\}, \end{aligned}$$

But that still leaves us with the ordered pair and the mapping notation to deal with.

Further Scholium on Classes (cont.)

(4) In Kunen's discussion of transfinite induction, he points out that the non-empty class $\mathbf{C} \subseteq \mathbf{ON}$ for which we prove a least element may involve additional parameters, i.e., other free variables, whereas $\mathbf{ON}$ contains only one free variable. More explicitly, if $\mathbf{C}(x, z_1, \ldots, z_n)$ is a wff with exactly $x, z_1, \ldots, z_n$ free, then the universal closure of the following is a theorem:

$$(\forall x(\mathbf{C} \to \mathbf{ON}) \land \exists x \mathbf{C}) \to \exists x(\mathbf{C} \land \forall y(\mathbf{C}(y, z_1, \ldots, z_n) \to x \leq y).$$

Two puzzles:

- For arbitrary classes $\mathbf{C}$ and $\mathbf{C}'$ with not all the same free variables, what does $\mathbf{C} \subseteq \mathbf{C}'$ mean?
- Kunen is using (or more gently) inviting an ambiguous use of the bold notation, viz., for the wff $\mathbf{C}$ and its extension, viz., $\mathbf{C} = \{x \mid \mathbf{C}\}$.

Policy re Classes

We can proceed as follows. First, we don't want to conflate a predicate with its extension, i.e., a syntactic with a semantic entity. I think we can find our way clear if we go back to an idea introduced earlier in connection with recognizing when the Comprehension Scheme is satisfied in a given structure $\mathfrak{A}$ for the language of ZF. There we said that an $n+1$-ary relation R on $|\mathfrak{A}|$ is definable in $\mathfrak{A}$ iff there is a wff $\varphi(v_0, \ldots, v_n)$ having exactly the first $n+1$ variables $v_0, \ldots, v_n$ free such that

$$R = \{\langle a_0, \ldots, a_n\rangle : \models_{\mathfrak{A}} \varphi \, [\![a_0, \ldots, a_n]\!]\}.$$

Terminology. By a *class-defining wff* we mean a wff C s.t. $FV(C) = \{v_0, \ldots, v_n\}$ for some n. Then the extension $\mathbf{C}_{\mathfrak{A}}$ of C in $\mathfrak{A}$ is

$$\mathbf{C}_{\mathfrak{A}} = \{\langle a_0, \ldots, a_n\rangle \mid \models_{\mathfrak{A}} C \, [\![a_0, \ldots, a_n]\!]\}.$$

When $\mathfrak{A}$ is understood to be arbitrary, we can drop the subscript, and with some sloppiness write $\mathbf{C} = \{\langle x_0, \ldots, x_n\rangle \mid C(x_0, \ldots, x_n)\}$.

Policy re Classes (cont.)

This gives us *two* distinct ways of talking about classes:

1. syntactically in terms wff's involving class defining expressions, and
2. semantically in terms of (meta) set operations on extensions in a generic structure for the language.

Unfortunately, (i) when we move from predicates to their extensions, we loose in the notational information about the arity of the class defined, and (ii) we need to generalize the notion of an extension to a *parameterized* extension so that we can perform (meta) set operations on classes of different arities. So let $\mathit{Arity}(\mathbf{C})$ indicate the arity of $\mathbf{C}$. And let's first try our hand at translation from class notation into the object language.

Policy re Classes (cont.)

Translation Manual.

- $x \in \mathbf{C}$ becomes $C(x, v_1, \ldots, v_n)$ if $\mathit{Arity}(\mathbf{C}) = n + 1$.
- $\mathbf{C_1} \subseteq \mathbf{C_2}$ becomes $\forall v_0 \cdots \forall v_k(C_1 \to C_2)$, where $k = \min\{\mathit{Arity}(\mathbf{C_1}), \mathit{Arity}(\mathbf{C_2})\} - 1$.
- $\mathbf{C_1} \cap \mathbf{C_2}$ becomes $(C_1 \wedge C_2)$.
- $x \cap \mathbf{C}$ becomes $\{y \in x \mid C(x, v_1, \ldots, v_n)\}$ if $\mathit{Arity}(\mathbf{C}) = n + 1$ and y is the first variable distinct from $x, v_1, \ldots, v_n$.
- $\mathbf{C_1} \times \mathbf{C_2}$ becomes $(C_1(v_0, \ldots, v_n) \wedge C_2(v_{n+1}, \ldots, v_{n+m+1}))$ if $\mathit{Arity}(\mathbf{C_1}) = n$ and $\mathit{Arity}(\mathbf{C_2}) = m$.
- Suppose $\mathit{Arity}(\mathbf{F}) = n + 2$, $\mathit{Arity}(\mathbf{C}) = n + 1$, and $\mathit{Arity}(\mathbf{D}) = 1$. Then $\mathbf{F} : \mathbf{C} \to \mathbf{D}$ becomes

$$\forall v_0 \cdots \forall v_n(C(v_0, \ldots, v_n) \to \exists! v_{n+1}(F(v_0, \ldots, v_{n+1}) \wedge D(v_{n+1})).$$

Policy re Classes (cont.)

Translation Manual (cont.). Finally, if $\mathbf{F} : \mathbf{C} \to \mathbf{D}$ s.t. $\mathbf{F}$ has arity $n+2$, and $a_0, \ldots, a_n$ are sets, we want $\mathbf{F} \restriction (a_0 \times \cdots \times a_n)$ to be the function with domain $a_0 \times \cdots \times a_n$ s.t. for any $(n+1)$-tuple $\langle x_0, \ldots, x_n \rangle \in (a_0 \times \cdots \times a_n)$,

$$\mathbf{F} \restriction (a_0 \times \cdots \times a_n)(x_0, \ldots, x_n) = y$$

iff $F(x_0, \ldots, x_n, y)$.

Example. Let $S(v_0, v_1)$ be the wff $v_0 \cup \{v_0\} = v_1$. Then $\mathbf{S} : \mathbf{V} \to \mathbf{V}$, and

$$\mathbf{S} \restriction \omega = \{\langle x, y \rangle \mid x \in \omega \ \wedge \ x \cup \{x\} = y\}.$$

Example. Let $UN(v_0, v_1, v_2)$ be the wff $v_0 \cup v_1 = v_2$. Then $\mathbf{UN} : \mathbf{V} \times \mathbf{V} \to \mathbf{V}$, and for (any) sets a and b

$$\mathbf{UN} \restriction (a \times b) = \{\langle x, y, z \rangle \mid \langle x, y \rangle \in (a \times b) \ \wedge \ x \cup y = z\}.$$

Transfinite Recursion

Recursion Theorem Schema. If $\mathbf{F} : \mathbf{V} \to \mathbf{V}$, then there is a $\mathbf{G} : \mathbf{ON} \to \mathbf{V}$ unique (up to extensional equivalence) s.t. for all α

$$\mathbf{G}(\alpha) = \mathbf{F}(\mathbf{G} \restriction \alpha).$$

Pf. *Uniqueness*. Suppose $\mathbf{G}_1$ and $\mathbf{G}_2$ both meet the condition, but disagree on some α. Let β be the least ordinal on which they disagree. Then $\mathbf{G}_1 \restriction \beta = \mathbf{G}_2 \restriction \beta$, so

$$\begin{aligned} \mathbf{G}_1(\beta) &= \mathbf{F}(\mathbf{G}_1 \restriction \beta) \\ &= \mathbf{F}(\mathbf{G}_2 \restriction \beta) \\ &= \mathbf{G}_2(\beta), \end{aligned}$$

contradicting the assumption that they disagree on β.

Transfinite Recursion (cont.)

Existence. What's tricky is that for any wff $F(x, y)$ we need to construct a wff $G(\alpha, y)$ that has the desired property, i.e.,

$$\forall x \exists ! y F(x, y) \to \forall \alpha \exists ! y G(\alpha, y) \wedge \forall \alpha \exists x \exists y (x = \mathbf{G} \restriction \alpha \wedge F(x, y) \wedge G(\alpha, y)),$$

where

$$\mathbf{G} \restriction \alpha = \{\langle \beta, z \rangle \mid \beta < \alpha \wedge G(\beta, z)\}.$$

Step 1. Call a function g a δ-approximation just in case $dom(g) = \delta$ and

$$\forall \alpha < \delta \, [g(\alpha) = \mathbf{F}(g \restriction \alpha)].$$

Just to be clear, we are introducing an object language predicate $Approx(\delta, g)$ s.t.

$$\begin{aligned} \forall \delta \forall g (Approx(\delta, g) \quad \leftrightarrow \quad & [Fn(g) \ \wedge \ dom(g) = \delta \ \wedge \\ & \forall \alpha < \delta \, \forall y \, (\langle \alpha, y \rangle \in g \leftrightarrow F(g \restriction \alpha, y))]) \end{aligned}$$

Transfinite Recursion (Steps 2–4)

Step 2. We want to show that if g_1 is a δ_1-approximation and g_2 is a δ_2-approximation, then for all $\alpha < \delta_1 \cap \delta_2$, $g_1(\alpha) = g_2(\alpha)$. But this is patent. For suppose otherwise. Let α be the least ordinal s.t. $g_1(\alpha) \neq g_2(\alpha)$. Then $g_1 \restriction \alpha = g_2 \restriction \alpha$. Thus, $g_1(\alpha) = \mathbf{F}(g_1 \restriction \alpha) = \mathbf{F}(g_2 \restriction \alpha) = g_2(\alpha)$.

Step 3. We want to show that $\forall \delta \exists ! g \; \mathit{Approx}(\delta, g)$.
By transfinite induction:

- The empty function 0 is the unique 0-approximation.
- Suppose g is the unique δ-approximation. Then $g \cup \{\langle \delta, \mathbf{F}(g) \rangle\}$ is the unique $(\delta + 1)$-approximation.
- Suppose δ is a limit ordinal and $\forall \xi < \delta$ there exists a unique ξ-approximation g_ξ. Then $\bigcup \{ g_\xi \mid \xi < \delta \}$ is the unique δ-approximation.

Step 4. Now let $G(\alpha, y)$ be the wff

$$\exists g (\mathit{Approx}(\alpha + 1, g) \;\wedge\; \langle \alpha, y \rangle \in g).$$

Then $\mathbf{G}(\alpha) = g(\alpha)$, where g is the unique $\alpha + 1$ approximation. ■

Applications to Transfinite Recursion

Note that in the theorem $F(x, y)$ can be *any* function-like class. Let, e.g., $F(x, y)$ be $y = \{x\}$. Note that no matter what F is, $\mathbf{G} \restriction 0 = 0$. In this instance

$$\begin{array}{rcl}
\mathbf{G}(0) & = & \mathbf{F}(0) \\
 & = & \{0\} = 1. \\
\mathbf{G}(1) & = & \mathbf{F}(\mathbf{G} \restriction 1) \\
 & = & \mathbf{F}(\{\langle 0, 1\rangle\}) \\
 & = & \{\{\langle 0, 1\rangle\}\} \\
\mathbf{G}(2) & = & \mathbf{F}(\mathbf{G} \restriction 2) \\
 & = & \{\{\langle 0, 1\rangle, \langle 1, \{\{\langle 0, 1\rangle\}\}\rangle\}\} \\
 & \vdots & \\
\mathbf{G}(\omega) & = & \{\{\langle n, \{\mathbf{G} \restriction n\}\rangle \mid n \in \omega\}\},
\end{array}$$

and so on.

Applications to Transfinite Recursion (cont.)

This didn't give us transfinite recursion in the form we expected, viz.,

$$\begin{aligned}
\mathbf{G}(0) &= \textit{something}\\
\mathbf{G}(S(\beta)) &= \mathbf{H}(\mathbf{G}(\beta), \beta)\\
\mathbf{G}(\gamma) &= \mathbf{H}'(\mathbf{G}\restriction\gamma, \gamma) \text{ if } \gamma \text{ a limit ordinal.}
\end{aligned}$$

OR in terms of two variables,

$$\begin{aligned}
\mathbf{G}(\alpha, 0) &= \mathbf{J}(\alpha)\\
\mathbf{G}(\alpha, S(\beta)) &= \mathbf{H}(\mathbf{G}(\alpha, \beta), \alpha, \beta)\\
\mathbf{G}(\alpha, \gamma) &= \mathbf{H}'(\mathbf{G}\restriction\gamma, \alpha, \gamma) \text{ if } \gamma \text{ a limit ordinal.}
\end{aligned}$$

The trick is that we have to define **F** piecemeal to cover the cases covered by **J**, **H**, and **H**′ (as well as the "garbage" case).

The Recursive Definition of Addition

Preliminary Defn.

$$\beta - 1 = \begin{cases} \beta & \text{if } \beta = 0 \text{ or } \beta \text{ is a limit ordinal} \\ \eta & \text{if } S(\eta) = \beta. \end{cases}$$

Now define $\mathbf{F}_\alpha(x)$, where α is just an additional parameter in $F(x, y)$ and $G(\beta, y)$ in the Recursion Theorem:

$$\mathbf{F}_\alpha(x) = \begin{cases} \alpha & \text{if } x = 0 \\ S(x(dom(x) - 1)) & \text{if } Fn(x) \wedge SuccON(dom(x)) \\ \bigcup\{x(\xi) \mid \xi < dom(x)\} & \text{if } Fn(x) \wedge LimitON(dom(x)) \\ 0 & \text{otherwise} \end{cases}$$

Then the unique $\mathbf{G}_\alpha : \mathbf{ON} \to \mathbf{V}$ given by the Recursion Theorem is s.t.

$$\begin{array}{rcl} \mathbf{G}_\alpha(\beta) & = & \alpha \\ \mathbf{G}_\alpha(S(\beta)) & = & S((\mathbf{G}_\alpha \restriction S(\beta))(\beta)) \\ \mathbf{G}_\alpha(\beta) & = & \sup\{(\mathbf{G}_\alpha \restriction \beta)(\xi) \mid \xi < \beta\} \text{ if } \beta \text{ a limit ordinal.} \end{array}$$

Recursion* Theorem

Theorem. Let $\mathbf{R}(\alpha,\beta)$, $\mathbf{J}(\alpha)$, $\mathbf{H}(x,\alpha,\beta)$, and $\mathbf{H}'(x,\alpha,\gamma)$ be function-like classes such that for all α, β, and γ,

$$\begin{aligned}
\mathbf{R}(\alpha,0) &= \mathbf{J}(\alpha)\\
\mathbf{R}(\alpha,S(\beta)) &= \mathbf{H}(\mathbf{R}(\alpha,\beta),\alpha,\beta)\\
\mathbf{R}(\alpha,\gamma) &= \mathbf{H}'(\mathbf{R}\restriction\gamma,\alpha,\gamma) \text{ if } \gamma \text{ a limit ordinal.}
\end{aligned}$$

Then there is an extensionally unique $\mathbf{F}_\alpha : \mathbf{V} \to \mathbf{V}$ s.t.

$$\forall\alpha\forall\beta[\mathbf{R}(\alpha,\beta) = \mathbf{F}_\alpha(\mathbf{R}\restriction\beta)].$$

Pf. Hint: Define $\mathbf{F}_\alpha$ piecewise from $\mathbf{J}$, $\mathbf{H}$, and $\mathbf{H}'$ along with a "garbage" clause, using in the second case the function-like class $\beta - 1$, as well.

Ordinal Exponentiation

Although it's possible, though difficult, to give a purely combinatorial definition of ordinal exponentiation (see Kunen, Exercise 7, Chapt. 1.) it's far more intuitive is to define it by transfinite recursion as iterated multiplication.

Defn. α^β is define by transfinite recursion on β as follows.

- $\alpha^0 = 1$.
- $\alpha^{S(\beta)} = \alpha^\beta \cdot \alpha$.
- If β is a limit ordinal, then $\alpha^\beta = \sup\{\alpha^\xi \mid \xi < \beta\}$.

This fits the form of the Recursion* Theorem.

Cardinality Comparisons [$\mathrm{ZF}^{-}-\mathrm{Inf}-\mathrm{P}$]

Defn.

- $A \preceq B$ iff $\exists f : A \to B$ that is injective (1-1).
- $A \sim B$ iff $\exists f : A \to B$ that is bijective.
- $A \prec B$ iff $A \preceq B$ and $B \not\preceq A$.

Lemma.

- $\forall A \;\; A \sim A$
- $\forall A \forall B (A \sim B \to B \sim A)$
- $\forall A, B, C (A \sim B \to (B \sim C \to A \sim C))$

Lemma.

- $\forall A \;\; A \preceq A$
- $\forall A, B, C (A \preceq B \to (B \preceq C \to A \preceq C))$
- *Schröder-Bernstein Theorem*: $\forall A, B (A \preceq B \;\wedge\; B \preceq A \;\to\; A \sim B)$
- (AC) $\forall A, B (A \preceq B \;\vee\; B \preceq A)$

Introducing Cardinals [ZF$^-$−Inf−P]

Defn. Suppose $\langle A, R\rangle$ is a well-ordering. Then $\exists!\alpha$ *type*$(A, R) = \alpha$. Let $\mathbf{C}_A =_{df} \{\alpha \in \mathbf{ON} \mid A \sim \alpha\}$. Let $|A| = \min \mathbf{C}_A$.

What if A does not come well-ordered?
$|A| = \alpha$ means that A *can* be well-ordered and α is the least ordinal s.t. $A \sim \alpha$.

Lemma. (AC)

- $\forall A \exists! \alpha \;\; |A| = \alpha$
- $\forall A, B(A \sim B \leftrightarrow |A| = |B|)$

Defn. $Card(x)$ iff $ON(x) \wedge \; |x| = x$.

Thus, if $Card(\alpha)$ and $\beta < \alpha$, then $\beta \prec \alpha$.

Some Modest Lemmata

Lemma. If $|\alpha| \leq \beta \leq \alpha$, then $|\beta| = |\alpha|$.

Pf. Suppose $|\alpha| \leq \beta \leq \alpha$. Since $\beta \leq \alpha$, $\beta \subseteq \alpha$, and hence $\beta \preceq \alpha$ by the identity map. On the other hand, $\alpha \sim |\alpha| \preceq \beta$. So, $\alpha \preceq \beta$, and hence by Schröder-Bernstein, $\alpha \sim \beta$. Hence $|\alpha| = |\beta|$. ■

Lemma. $\forall n \in \mathbb{N},\ n \not\sim n + 1$.

Pf. Exercise. (Hint: Use induction.)

Lemma. $\forall n \in \mathbb{N}(n \neq 0 \to \forall m \in n(n \setminus \{m\} \sim n - 1))$.

Pf. Exercise.

Some Modest Lemmata (cont.)

Lemma. $\forall\alpha(LimitON(\alpha) \to \forall n\ \alpha \not\sim n)$.
Pf. Suppose α is a limit ordinal and that $f : \alpha \to n$ is bijective. We argue by induction on n that $\{f^{-1}(m) \mid m < n\}$ is bounded in α. Let

$$\beta = S(\sup\{f^{-1}(m) \mid m < n\}).$$

It follows that $\beta \notin dom(f)$ yet $\beta \in \alpha$.

Lemma. $\forall n \in \mathbb{N}\ \forall\alpha \in \mathbf{ON}(\alpha \sim n \to \alpha = n)$.
Pf. Do induction on $\mathbb{N}$. The $n = 0$ case is immediate. Suppose then for arbitrary $n \in \mathbb{N}$ that $\forall\alpha \in \mathbf{ON}(\alpha \sim n \to \alpha = n)$. Now suppose $f : \beta \to n + 1$ is bijective. By the previous lemma, β is a successor ordinal. So, let γ be its immediate predecessor, and let $f' = f \restriction \gamma$. Then $ran(f') = (n + 1) \setminus \{f(\gamma)\}$. By an earlier lemma, $ran(f') \sim n$. Since f' is a bijection $\gamma \sim n$. By the inductive hypothesis, $\gamma = n$. Hence $\beta = S(\gamma) = S(n) = n + 1$. ■

Some Modest Lemmata (cont.)

Corollary. *Card*(n) for every $n \in \mathbb{N}$.

Pf. For any ordinal, and thus any natural number n, $|n| \sim n$. Immediately from the last lemma, $|n| = n$. Thus *Card*(n). ■

Corollary [ZF$^-$−P]. ω is a cardinal.

Pf. We want to show $\omega \not\sim \alpha$ for every $\alpha < \omega$. This is immediate since every such α is a natural number, and no natural number is equinumerous with a limit ordinal. ■

The Finite, the Countable, the Infinite

Defn [$\mathrm{ZF}^- - \mathrm{Inf} - \mathrm{P}$]. A is *finite* iff $A \sim n$ for some $n \in \mathbb{N}$. Otherwise A is *infinite*.

Defn [$\mathrm{ZF}^- - \mathrm{P}$]. A is *countable* iff A is finite or $A \sim \omega$. Otherwise A is *uncountable*.

Defn [$\mathrm{ZF}^- - \mathrm{Inf} - \mathrm{P}$]. A is *Dedekind infinite* iff $A \sim B$ for some proper $B \subset A$. Else A is *Dedekind finite*.

Lemma [$\mathrm{ZF}^- - \mathrm{P}$]. Equivalents of Dedekind infinite:

- $\exists f : A \to A$ that is injective but not surjective.
- A has a countably infinite subset.
- $\exists f : \omega \to A$ that is injective.

Relations of the Above

[ZF$^-$−Inf−P]

- Dedekind infinite $\Rightarrow$ infinite. Thus, contrapositively,
- finite $\Rightarrow$ Dedekind finite

[ZF$^-$−P]

- AC $\vdash \forall x(x$ is infinite $\leftrightarrow x$ is Dedekind infinite), and thus
- AC $\vdash \forall x(x$ is finite $\leftrightarrow x$ is Dedekind finite).

In fact less than AC is needed. The axiom of countable choice AC_ω suffices, where

$\mathbf{AC}_\omega$: Every countable set of non-empty sets has a choice function.

Cardinal Addition and Multiplication [ZF$^-$−Inf−P]

Convention. Let $\kappa, \lambda, \ldots$ range over cardinals.

Defn. $\kappa \oplus \lambda = |\, \kappa \times \{0\} \cup \lambda \times \{1\} \,|$.

Defn. $\kappa \otimes \lambda = |\, \kappa \times \lambda \,|$.

Lemma. Cardinal addition and multiplication are both associative and commutative.

Pf. Exercise.

Lemma [ZF$^-$−P].
$\forall m, n < \omega\ (m \oplus n = m + n < \omega\ \wedge\ m \otimes n = m \cdot n < \omega)$.

Pf. Exercise.

Cardinal Addition and Multiplication (cont.)

Lemma. For any $\alpha \geq \omega$, $|\, \alpha + 1 \,| = |\, 1 + \alpha \,|$.

Pf. Exercise.

Lemma. Every infinite cardinal is a limit ordinal.

Pf Suppose $\kappa \geq \omega$ but $\kappa = \beta + 1$ for some β. Then

$$|\kappa| = |\, \beta + 1 \,| = |\, 1 + \beta \,| = |\beta| \leq \beta < \kappa,$$

which entails that κ is not a cardinal. ■

Cardinal Addition and Multiplication (cont.)

Theorem. If κ is infinite, then $\kappa \otimes \kappa = \kappa$.

Pf. By transfinite induction. We show that if the statement holds for all cardinals $< \kappa$, then it holds for κ. So suppose it holds for all cardinals $< \kappa$. Then for all $\alpha < \kappa$, $|\, \alpha \times \alpha \,| = |\alpha| \otimes |\alpha| < \kappa$, where we rely on the second to last lemma to cover the case where α is finite.
Now let $\lhd$ be the well-ordering of $\kappa \times \kappa$ such that $\langle \alpha, \beta \rangle \lhd \langle \gamma, \delta \rangle$ iff

$$\max(\alpha, \beta) < \max(\gamma, \delta)$$

or

$$\max(\alpha, \beta) = \max(\gamma, \delta) \ \wedge \ \langle \langle \alpha, \beta \rangle, \langle \gamma, \delta \rangle \rangle \in L,$$

where L is the lexicographic order on $\kappa \times \kappa$.

Proof of the Theorem (cont.)

Let $\xi = \max(\alpha, \beta) + 1$. Each $\langle \alpha, \beta \rangle \in \kappa \times \kappa$ has at most $|\, \xi \times \xi \,|$ predecessors in $\lhd$, where $\xi < \kappa$. By the inductive hypothesis then,

$$\begin{aligned} |\, \xi \times \xi \,| &= |\xi| \otimes |\xi| \\ &< \kappa. \end{aligned}$$

Hence *type*$(\kappa \times \kappa, \lhd) \leq \kappa$, and so $|\, \kappa \times \kappa \,| \leq \kappa$. But obviously $\kappa \leq |\, \kappa \times \kappa \,|$. Therefore $|\, \kappa \times \kappa \,| = \kappa$. ■

Some Corollaries

Cor. If either κ or λ is infinite and neither is zero, then $\kappa \oplus \lambda = \kappa \otimes \lambda = \max(\kappa, \lambda)$.
Pf. $\max(\kappa, \lambda) \leq \kappa \oplus \lambda \leq \kappa \otimes \lambda \leq \max(\kappa, \lambda)$. ■

Cor. If κ is an infinite cardinal, then $|\kappa^{<\omega}| = \kappa$.
Pf. By the theorem, there is a bijection $g : \kappa \times \kappa \to \kappa$.
Define by recursion for each positive integer n a function $f_n : \kappa^n \to \kappa$ as follows

$$\begin{aligned} f_1(\alpha_1) &= \alpha_1 \\ f_{n+1}(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}) &= g(f_n(\alpha_1, \ldots, \alpha_n), \alpha_{n+1}) \end{aligned}$$

By Replacement $\{\kappa^n \mid n \in \omega\}$ is a set. Let $f : \bigcup_n \kappa^n \to \omega \times \kappa$ be defined:

$$f(\langle \alpha_1, \ldots, \alpha_n \rangle) = \langle n, f_n(\alpha_1, \ldots, \alpha_n) \rangle.$$

Thus $|\kappa^{<\omega}| = |\bigcup_n \kappa^n| = |\omega \times \kappa| = \kappa$. ■