Reflection Theorems

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Set Models of ZFC?

Observation. Recall that for any regular $\kappa > \omega$, the hereditary set $H(\kappa)$ is a model of ZFC $-$ P, i.e., for each axiom σ of ZFC $-$ P,

$$\text{ZFC} \vdash \forall k > \omega(\kappa \text{ is regular} \rightarrow \sigma^{H(\kappa)}).$$

And, moreover, ZFC $\vdash \exists\kappa > \omega(\kappa \text{ is regular})$. So we know there are set models of ZFC $-$ P.

Question. Can we find a set model of all of ZFC? Recall we had that, for any axiom σ of ZFC,

$$\text{ZFC} \vdash \forall\kappa(Si(\kappa) \rightarrow \sigma^{H(\kappa)}).$$

But, unfortunately, ZFC $\nvdash \exists\kappa\, Si(\kappa)$. What we need is a set A s.t. ZFC $\vdash \sigma^A$ for each axiom σ of ZFC.

Is there such a set A?

Reflection Theorems

Theorem. Suppose we have a hierarchy of sets $Z(\alpha)$ defined by recursion s.t.
(1) $\alpha < \beta \to Z(\alpha) \subset Z(\beta)$,
(2) If γ is a limit ordinal, then $Z(\gamma) = \bigcup_{\xi<\gamma} Z(\xi)$, and
(3) $\mathbf{Z} = \bigcup_{\alpha \in \mathbf{ON}} Z(\alpha)$.
Then for any wff φ

$$\mathrm{ZF} \vdash \forall\alpha(\exists\beta > \alpha)(\varphi \text{ is absolute for } Z(\beta), \mathbf{Z}).$$

■

Taking $Z(\alpha) = R(\alpha)$ and $\mathbf{Z} = \mathbf{WF}$, we have:

Corollary (Reflection Theorem). For any wff φ,

$$\mathrm{ZF} \vdash \forall\alpha(\exists\beta > \alpha)(\varphi \text{ is absolute for } R(\beta)),$$

i.e.,

$$\mathrm{ZF} \vdash \forall\alpha(\exists\beta > \alpha)(\varphi^{R(\beta)} \leftrightarrow \varphi).$$

Further Corollaries

Corollary. For any sentences $\sigma_1, \ldots, \sigma_n$,

$$\text{ZF} \vdash \forall\alpha(\exists\beta > \alpha)\left(\bigwedge_{i=1}^{n}(\sigma_i^{R(\beta)} \leftrightarrow \sigma_i)\right). \quad \blacksquare$$

Corollary. For any axiomatic extension S of ZF, if $\sigma_1, \ldots, \sigma_n$ are among the axioms of S, then

$$\text{S} \vdash \forall\alpha(\exists\beta > \alpha)\left(\bigwedge_{i=1}^{n}\sigma_i^{R(\beta)}\right). \quad \blacksquare$$

Taking S = ZFC, this tells us that any *finite* subset of the axioms of ZFC has a set model, viz., $R(\beta)$ for some ordinal β. However,

Corollary. Let S be any set of axioms extending ZF, and $\varphi_1, \ldots, \varphi_n$ any axioms of S. If from $\varphi_1, \ldots, \varphi_n$ one can prove all axioms of S, then S is inconsistent.

Proof of the Last Corollary

Proof . Suppose $\sigma_1, \ldots, \sigma_n$ is a finite axiomatization of ZF, and take S to be ZF in the last corollary. Thus,

$$\mathsf{ZF} \vdash \forall\alpha(\exists\beta > \alpha)\left(\bigwedge_{i=1}^{n} \sigma_i^{R(\beta)}\right).$$

From this it is clear that

$$\mathsf{ZF} \vdash \exists x\left(ON(x) \wedge \bigwedge_{i=1}^{n} \sigma_i^{R(x)}\right).$$

Let β be such an ordinal, in fact, the least such. Then

$$\mathsf{ZF} \vdash \textstyle\bigwedge_{i=1}^{n} \sigma_i^{R(\beta)},$$

and so $R(\beta)$ is a model of ZF, in fact a transitive model.

Proof (cont.)

It follows from an earlier lemma that if $\alpha \in R(\beta)$, then

$$R(\alpha)^{R(\beta)} = R(\alpha) \cap R(\beta) = R(\alpha),$$

and thus $R(\cdot)$ is absolute for $R(\beta)$. Hence

$$\mathsf{ZF} \vdash (\exists x \in R(\beta)) \left(ON(x) \wedge \bigwedge_{i=1}^{n} \sigma_i^{R(x)} \right).$$

But since by hypothesis β was chosen to be the least ordinal s.t. $R(\beta)$ models ZF, there can be no ordinal $\alpha \in R(\beta)$ s.t. $\bigwedge_{i=1}^{n} \sigma_i^{R(\alpha)}$. So ZF is inconsistent. $\blacksquare$.

Why ZF is not Finitely Axiomatizable (if consistent)

Corollary. ZF is not finitely axiomatizable if consistent.

Proof. Suppose ZF consistent and let S_0 be any proposed finite axiomatization. Then $S_0 \vdash \sigma$ for each axiom σ of ZF. In the last corollary, take $S = S_0 \cup \text{ZF}$. Trivially $S_0 \vdash \varphi$ for any $\varphi \in S_0$. Thus, $S_0 \vdash \psi$ for any $\psi \in S$. By the corollary, it follows that S, and thus ZF, is inconsistent. ■

Countability Results

Theorem (AC). Take $\mathbf{Z}$ to be any class and $\varphi_1, \ldots, \varphi_n$ any wffs. Then for any $X \subseteq \mathbf{Z}$ there exists an A such that

1. $X \subseteq A \subseteq \mathbf{Z}$,
2. $\varphi_1, \ldots, \varphi_n$ are absolute for $A, \mathbf{Z}$, and
3. $|A| \leq \max(\omega, |X|)$.

Lemma. There is a transitive set M having the properties of A above.

Proof. Use the Mostowski collapsing isomorphism.

Corollary. Suppose that S is a set of axioms extending ZFC and $\sigma_1, \ldots, \sigma_n$ any axioms of S. Then it is provable from S that there exists a countably infinite, transitive set M s.t. $\sigma_1, \ldots, \sigma_n$ are true in M.