The Constructible Universe **L**

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On Definable Subsets of a Set

- Given a set A and a wff $\varphi(x)$ in one free variable, by Extensionality and Comprehension, φ defines a subset of A:

$$\{x \in A \mid \varphi(x)\}.$$

- Note that it may be that not all subsets of A are definable. For if A is infinite, $\mathscr{P}(A)$ is uncountable, but there are only countably many wff's in $\mathcal{L}_{ZF}$.
- This gives us a hint of the sort of strategy used to construct **L**. Suppose, at each stage of building the cumulative hierarchy, we were to take not the *power set* of the current stage to produce the next stage, but instead only the set of all *definable* subsets of the current stage.
- And, at least at infinite stages, the resulting set will be strictly smaller in cardinality than the full power set, one can hold out the hope that, e.g., 2^ω, becomes bounded by ω_1, so that **L** is a model of CH.

The Idea of $Df(A, n)$

For technical reasons, we want to make two modifications to this approach.

- First, we consider only the subsets of A definable by a wff *relativized to* A.
- Second, we want to consider not just these subsets of A, but also the subsets of A^n definable by wffs relativized to A with n free variables, so that we can consider at each stage of the construction of **L** the set of all subsets definable from a finite number of elements of that stage.

So what we are looking to define, for each $n > 0$, is the set of all sets of the form

$$\{\langle x_1, \ldots, x_n \rangle \in A^n \mid \varphi^A(x_1, \ldots, x_n)\}.$$

Call this intended set of sets $Df(A, n)$.

Toward a Rigorous Definition of $Df(A, n)$

- We're looking for a definition of $Df(A, n)$ in the object language, but we can't directly quantify over wffs of $\mathcal{L}_{ZF}$.
- So, one route would be to formalize the syntax of 1st-order logic in ZF, so that wff's can be identified with particular sets. But that would be extremely complicated. Fortunately, there is a direct algebraic characterization.
- To grease the wheels, we'll consider ${}^nA$ instead of A^n. Thus, we're looking to characterize the set of all sets of the form

$$\{s \in {}^nA \mid \varphi^A(s(0), \ldots, s(n-1))\},$$

  for wff's $\varphi(v_0, \ldots, v_{n-1})$.
- Further, to avoid clumsy notation, from now on well consider A^n to be just ${}^nA$.

Algebraic Characterization of $Df(A, n)$

Def. For $n \in \omega$ s.t. $n > 0$ and for $0 < i, j < n$ let:

$$Diag_{=}(A, n, i, j) =_{df} \{s \in A^n \mid s(i) = s(j)\}$$

$$Diag_{\in}(A, n, i, j) =_{df} \{s \in A^n \mid s(i) \in s(j)\}.$$

These correspond to atomic wffs. The idea now is to build by recursion using operations corresponding to negation, conjunction and existential quantification. Set intersection corresponds to conjunction and relative complementation to negation. For existential quantification we need a projection operation:

$$Proj(A, R, n) =_{df} \{s \in A^n \mid \exists t \in R : t \restriction n = s\}.$$

Thus, suppose $R \in Df(A, n+1)$ "in virtue of" $\varphi(v_0, \ldots, v_n)$. Then $Proj(A, R, n) \in Df(A, n)$ "in virtue of" $\exists v_n \varphi(v_0, \ldots, v_n)$.

Algebraic Characterization of $Df(A, n)$, (cont.)

Now, by recursion on $k \in \omega$ define a preliminary concept $Df'(k, A, n)$ for all n simultaneously, where k indicates the number of logical operators, as follows.

$$Df'(0, A, n) =_{df} \{Diag_{=}(A, n, i, j) \mid i, j < n\} \cup \{Diag_{\in}(A, n, i, j) \mid i, j < n\},$$

and

$$\begin{aligned} Df'(k+1, A, n) \quad =_{df} \quad & \{A^n \setminus R \mid R \in Df'(k, A, n)\} \cup \\ & \{R \cap S \mid R, S \in Df'(k, A, n)\} \cup \\ & \{Proj(A, R, n) \mid R \in Df'(k, A, n+1)\}. \end{aligned}$$

Then

$$Df(A, n) =_{df} \bigcup_{k \in \omega} Df'(k, A, n).$$

Adequacy Lemmas

Lemma. Let $\varphi(v_0, \dots, v_n)$ be a wff of $\mathcal{L}_{ZF}$. Then,

$$\forall A : \{s \in A^{n+1} : \varphi^A(s(0), \dots, s(n))\} \in Df(A, n+1).$$

Proof. By straightforward induction on the shape of φ. ■

Aside. There's something wrong in Kunen's presentation here. This is supposed to be a lemma schema. The proof is obviously in the metalanguage. But Kunen intimates that the displayed sentence is a theorem of ZF. So, how it is to be proved remains a mystery. My suspicion is it's not provable in ZF, nor does it need to be proven in ZF, any more than the following.

(Meta-)Lemma For every definable A and every $R \in Df(A, n)$ there is a wff $\varphi(v_0, \dots, v_{n-1})$ s.t.

$$R = \{s \in A^n \mid \varphi(s(0), \dots, s(n-1))\}. \quad \square$$

Enumerability of $Df(A, n)$

The second adequacy lemma establishes in the metalanguage that $Df(A, n)$ is countable. But this is something one wants to be able to prove in the object language.

Def. $En(\cdot, A, n) : \omega \to Df(A, n)$ is defined case-wise as follows.

- If $m = 2^i \cdot 3^j$ and $i, j < n$, then $En(m, A, n) = Diag_\epsilon(A, n, i, j)$.
- If $m = 2^i \cdot 3^j \cdot 5$ and $i, j < n$, then $En(m, A, n) = Diag_=(A, n, i, j)$.
- If $m = 2^i \cdot 3^j \cdot 5^2$, then $En(m, A, n) = A^n \setminus En(i, A, n)$.
- If $m = 2^i \cdot 3^j \cdot 5^3$, then $En(m, A, n) = En(i, A, n) \cap En(j, A, n)$.
- If $m = 2^i \cdot 3^j \cdot 5^4$, then $En(m, A, n) = Proj(A, En(i, A, n + 1), n)$.
- Otherwise $En(m, A, n) = 0$.

Results about $En(\cdot, A, n)$ and $Df(A, n)$

Lemma. For any n, $En(\cdot, A, n)$ is onto.

Corollary. $|Df(A, n)| \leq \omega$.

Lemma. Both Df and En are absolute for transitive models of $\mathrm{ZF} - \mathrm{P}$.

The Ordinal Definable Sets **OD**

Intuitively, a set A is definable in ZF from some finite sequence of ordinals $\alpha_0, \ldots, \alpha_n$ just in case there is a wff $\varphi(v_0, \ldots, v_{n+1})$ s.t.

$$\text{ZF} \vdash \forall x(\varphi(\alpha_0, \ldots, \alpha_n, x) \leftrightarrow x = A).$$

(N.B. We should expect that **ON** $\subseteq$ **OD** by taking φ to be $v_0 = x$.)

We have the same problem as with $Df(A, n)$ of appearing to need to quantify over wff's in order to define **OD** in the object language. But there is a trick, due to the fact that if $\beta > \max(\alpha_0, \ldots, \alpha_n, rank(A))$, then A is ordinal definable within $R(\beta)$, i.e.,

$$\text{ZF} \vdash (\forall x \in R(\beta))(\varphi(\alpha_0, \ldots, \alpha_n, x) \leftrightarrow x = A)^{R(\beta)}.$$

This follows from an instance of the Reflection Theorem, viz., that for any wff θ

$$\text{ZF} \vdash \forall \alpha(\exists \beta > \alpha)(\theta \text{ is absolute for } R(\beta)).$$

Object Language Definition of **OD**

Def. **OD** is the class of all sets A s.t.

$$(\exists \beta > rank(A))\ (\exists n \in \omega)\ (\exists s \in \beta^n)\ (\exists R \in Df(R(\beta), n+1)) \text{ s.t.}$$

$$(\forall x \in R(\beta))(s^\frown \langle x \rangle \in R \leftrightarrow x = A). \qquad \square$$

Adequacy Theorem. For any wff $\varphi(v_0, \ldots, v_{n+1})$,

$$\forall \alpha_0 \cdots \forall \alpha_n \forall A(\forall x(\varphi(\alpha_0, \ldots, \alpha_n, x) \leftrightarrow x = A) \rightarrow A \in \mathbf{OD}).$$

Proof. Basically, using the Reflection Theorem as informally rehearsed.
■

That's "half" of what needs to be shown for adequacy. The other "half" is to show that if $A \in \mathbf{OD}$, then there's some defining wff φ. It will turn out that there's a single $\varphi(v_0, v_1)$ that suffices for all $A \in \mathbf{OD}$, and thus that serves to map **ON** onto **OD**. To get there, we need a 1-1 map of **ON** onto $\mathbf{ON}^{<\omega}$.

$Enon : \mathbf{ON} \to \mathbf{ON}^{<\omega}$

The first step is to define a well ordering of $\mathbf{ON}^{<\omega}$.

Def. For all $s, t \in \mathbf{ON}^{<\omega}$ let $s \lhd t$ iff

- $\max(ran(s)) < \max(ran(t))$, or
- $\max(ran(s)) = \max(ran(t))$ and $dom(s) < dom(t)$, or
- $\max(ran(s)) = \max(ran(t))$ and $dom(s) = dom(t)$ and $\exists k \in dom(s)(s \restriction k = t \restriction k \ \wedge \ s(k) < t(k))$.

The last clause says that if s and t have the same length and the same maximum value, then compare them lexicographically.

Def. $Enon(\gamma)$ is the γ-th element of $\mathbf{ON}^{<\omega}$ in the order $\lhd$.

Lemma. $Enon$ is a 1-1 map of $\mathbf{ON}$ onto $\mathbf{ON}^{<\omega}$.

$Enod$: ON → OD

Def. $Enod$: **ON** → **OD** is defined case-wise as follows. For each $\gamma \in$ **ON**,

- If $Enon(\gamma) = s^\frown\langle\beta, n, m\rangle$, where $n, m \in \omega$, $s \in \beta^{<\omega}$, $dom(s) = n$, and for some (unique) $A \in R(\beta)$,

  $$(\forall x \in R(\beta))(s^\frown\langle x\rangle \in En(m, R(\beta), n+1) \leftrightarrow x = A),$$

  then $Enod(\gamma) = A$.
- Else $Enod(\gamma) = 0 \quad \square$.

Lemma. $Enod$ is onto.

The $\varphi(v_0, v_1)$ promised above is: $(v_0 \in \mathbf{ON} \ \wedge \ Enod(v_0) = v_1)$. Hence,

$$\mathrm{ZF} \ \vdash (\forall A \in \mathbf{OD})\ (\exists\alpha)\ (\forall x)\ (\varphi(\alpha, x) \leftrightarrow x = A).$$

Lemma. **ON** ⊆ **OD** and

$$(\forall x, y \in \mathbf{OD})(\bigcup x \in \mathbf{OD} \ \wedge \mathcal{P}(x) \in \mathbf{OD} \ \wedge \{x, y\} \in \mathbf{OD}).$$

Interlude: Some Relative Consistency Results

Extensionality fails in **OD** unless $\mathbf{V} = \mathbf{OD}$, which, as Kunen says, is "unlikely". (We put off the relative consistency of $\mathbf{V} = \mathbf{OD}$.) One wants to use rather the following proper subclass to get a model of ZF.

Def. $\mathbf{HOD} =_{df} \{x \in \mathbf{OD} \mid tr\,cl(x) \subset \mathbf{OD}\}$.

Lemma. $\mathbf{ON} \subset \mathbf{HOD} \subset \mathbf{OD}$ and **HOD** is transitive.

Lemma. $\forall x(x \in \mathbf{OD} \wedge x \subset \mathbf{HOD} \to x \in \mathbf{HOD})$.

Lemma. $\forall \alpha : (R(\alpha) \cap \mathbf{HOD}) \in \mathbf{HOD}$.

Theorem. ZF $\vdash$ **HOD** is a model of ZFC.

Corollary. Con(ZF) $\to$ Con(ZFC).

The Definable Power Set Operator $\mathcal{D}$

This is what we'll use in place of $\mathcal{P}$ to iteratively construct **L**.

Def. $\mathcal{D}(A)$ is defined to be the set of subsets $X \subseteq A$ s.t.

$$(\exists n \in \omega)\ (\exists s \in A^n)\ (\exists R \in Df(A, n+1)) : X = \{x \in A \mid s^\frown\langle x\rangle \in R\}. \ \square$$

Characterization Lemma. For any wff $\varphi(v_0, \ldots, v_n)$

$$\forall y_0 \ldots y_{n-1} \in A : \{x \in A \mid \varphi^A(y_0, \ldots, y_{n-1}, x)\} \in \mathcal{D}(A).$$

Lemma. For any A,

- $\mathcal{D}(A) \subseteq \mathcal{P}(A)$,
- if A is transitive, then $A \subseteq \mathcal{D}(A)$,
- $\forall X \subseteq A(|X| < \omega \to X \in \mathcal{D}(A))$, and
- (AC) $|A| \geq \omega \to |\mathcal{D}(A)| = |A|$.

The Constructible Universe

Def. By transfinite recursion on **ON** define $L(\alpha)$ as follows.

- $L(0) = 0$.
- $L(\alpha + 1) = \mathcal{D}(L(\alpha))$.
- If α is a limit ordinal, then $L(\alpha) = \bigcup_{\xi<\alpha} L(\xi)$.

Def. $\mathbf{L} = \bigcup_{\alpha \in \mathbf{ON}} L(\alpha)$.

Lemma. For all α,

- $L(\alpha)$ is transitive, and
- $\forall \xi < \alpha : L(\xi) \subseteq L(\alpha)$.

Def. For all $x \in \mathbf{L}$, $\rho(x) =$ the least α s.t. $x \in L(\alpha + 1)$. This is called the **L-rank** of x.

Lemma. $\forall \alpha : L(\alpha) = \{x \in \mathbf{L} \mid \rho(x) < \alpha\}$.

Lemma. Forall α

- $\alpha \in L(\alpha + 1)$ and $\rho(\alpha) = \alpha$.
- $L(\alpha) \cap \mathbf{ON} = \alpha$.

More Results about $\mathbf{L}$

Lemma. $\forall\alpha : L(\alpha) \in L(\alpha+1)$.

Lemma. $\forall\alpha : L(\alpha) \subseteq R(\alpha)$.

Lemma. Every finite subset of $L(\alpha)$ is in $L(\alpha+1)$.

Lemma. $\forall n \in \omega : L(n) = R(n)$.

Lemma. $L(\omega) = R(\omega)$.

Lemma (AC). $\forall\alpha \geq \omega : |L(\alpha)| = |\alpha|$.

Lemma (AC). $\forall\alpha > \omega\ (|L(\alpha)| = |R(\alpha)| \leftrightarrow\ \alpha = \beth_\alpha)$.

Scholium. $\mathcal{P}(\omega) \subseteq R(\omega+1)$, but there is no obvious reason why it must be the case that $\mathcal{P}(\omega) \subset \mathbf{L}$. In fact, it is later shown that it is consistent that $\mathcal{P}(\omega) \not\subset \mathbf{L}$, in which case $L(\alpha) \neq R(\alpha)$ for every $\alpha > \omega$. However, it will be shown that $\mathbf{V} = \mathbf{L}$ is consistent, in which case $L(\alpha) = R(\alpha)$ iff $\alpha = \beth_\alpha$.

Theorem. ZF $\vdash$ $\mathbf{L}$ is a model of ZF.

Some Relative Consistency Results

Def. The **Axiom of Constructibility** is the statement $\mathbf{V} = \mathbf{L}$.

Lemma. $L(\alpha)$ is absolute for transitive models of ZF $-$ P.

Theorem. ZF $\vdash$ ($\mathbf{L}$ is a model of ZF $+$ $\mathbf{V} = \mathbf{L}$).

Corollary. Con(ZF) $\rightarrow$ Con(ZF $+$ $\mathbf{V} = \mathbf{L}$)

Theorem (*Minimality of* $\mathbf{L}$). If $\mathbf{M}$ is any transitive proper class model of ZF $-$ P, then $\mathbf{L} = \mathbf{L}^{\mathbf{M}} \subseteq \mathbf{M}$.

More Minimality Results and Some Uniqueness Results

Folklore. There is a finite conjunction φ of axioms of ZF – P so that the notions of ordinal, rank, and $L(\alpha)$ are absolute for transitive models of φ.

Corollary. For any class $\mathbf{M}$:

$$\text{ZF} - \text{P} \vdash \text{If } \mathbf{M} \text{ is a transitive proper class and } \varphi^{\mathbf{M}}\text{, then } \mathbf{L} \subseteq \mathbf{M}.$$

Defn. For any *set* M, $o(M) =_{df} M \cap \mathbf{ON}$.

Lemma. If M is a transitive set, then $o(M)$ is the least ordinal not in M.

Theorem. The above finite conjunction φ of axioms of ZF – P is s.t.

$$\text{ZF} \vdash \forall M\ (\text{if } M \text{ is transitive and } \varphi^{M}\text{, then } L(o(M)) = \mathbf{L}^{M} \subseteq M).$$

Theorem. From φ above conjoined with $\mathbf{V} = \mathbf{L}$ it is provable that:

- If $\mathbf{M}$ is a transitive proper class, then $\mathbf{M} = \mathbf{L}$.
- $\forall M(\mathit{trans}(M) \wedge \varphi^{M} \rightarrow M = L(o(M)))$.

Minimality and Uniqueness (cont.)

Scholium. It follows from the above that any transitive model for ZF + $\mathbf{V} = \mathbf{L}$ is s.t.

(a) if it is a proper class $\mathbf{M}$, then $\mathbf{M} = \mathbf{L}$, and

(b) if it is a set M, then $M = L(\alpha)$ where $\alpha = o(M)$.

Theorem. ZF $\vdash (\mathbf{V} = \mathbf{L} \rightarrow \mathbf{V} = \mathbf{HOD})$.

Corollary. Con(ZF) $\rightarrow$ Con(ZF + $\mathbf{V} = \mathbf{HOD}$).

Corollary. ZF $\vdash \mathbf{V} = \mathbf{L} \rightarrow$ AC.

Scholium. There is a direct construction that well-orders each $L(\alpha)$ that does not rely on **HOD**, but this is too involved to develop at the moment.

CH and GCH in $\mathbf{L}$

CH follows from the fact that each subset of ω is constructed, not at stage $L(\omega)$, but at some stage $L(\alpha)$ for countable α so that $\mathcal{P}(\omega) \subseteq L(\omega_1)$. But since $|L(\omega_1)| = \omega_1$ as we saw earlier, it follows that $2^\omega \leq \omega_1$, thus cinching CH. More generally:

Theorem. If $\mathbf{V} = \mathbf{L}$, then for all infinite ordinals α, $\mathcal{P}(L(\alpha)) \subseteq L(\alpha^+)$.

Corollary. ZF $\vdash \mathbf{V} = \mathbf{L} \rightarrow$ AC + GCH.

Corollary. ZF $\vdash$ (AC + GCH)$^{\mathbf{L}}$.

Corollary. Con(ZF) $\rightarrow$ Con(ZFC + GCH).

Some Final Results

Theorem (ZF). ZF + $\mathbf{V} = \mathbf{L}$ has a countable transitive set model.

Lemma. Suppose $\mathbf{V} = \mathbf{L}$. Then $L(\kappa) = H(\kappa)$ if $\kappa > \omega$ and κ is regular.

Corollary (ZF). If $\kappa > \omega$ and κ is regular, then $L(\kappa)$ models ZF − P + $\mathbf{V} = \mathbf{L}$. Furthermore, if κ is weakly inaccessible, then $L(\kappa)$ models ZF + $\mathbf{V} = \mathbf{L}$.

Corollary. Con(ZF) $\rightarrow$ Con(ZFC + GCH + $\neg\exists\kappa : \kappa$ is weakly inaccessible).