

Exercise Set # 1

Due Tuesday, February 2.
(100 points: 10 each)

1. *Naïve Set Theory*. Although Cantor founded transfinite set theory, he never stated axioms. Is there a reason we need axioms? What is Russell's paradox? How does ZF^- avoid it? (Kunen throughout uses ZF^- for ZF without the Axiom of Foundation.)
2. **Defn.** A relation R on a class A is said to be
 - *reflexive* iff for all $x \in A$, $\langle x, x \rangle \in R$,
 - *symmetric* iff for all $x, y \in A$, if $\langle x, y \rangle \in R$, then $\langle y, x \rangle \in R$,
 - *transitive* iff for all $x, y, z \in A$, if $\langle x, y \rangle \in R$ and $\langle y, z \rangle \in R$, then $\langle x, z \rangle \in R$,
 - an *equivalence relation* on A iff R is reflexive, symmetric, and transitive on A .

Defn. Let R be an equivalence relation on A . Then

$$[x]_R =_{df} \{y \in A \mid \langle x, y \rangle \in R\}$$

is said to be the equivalence class of x under R on A .

Defn. A *partition* Π of a class A is a collection \mathcal{C} of non-empty subclasses of A s.t. (i) the members of \mathcal{C} are pairwise disjoint and (ii) $\bigcup \mathcal{C} = A$.

Establish the following.

Lemma. If R is an equivalence relation on A , then the collection $\{[x]_R \mid x \in A\}$ of equivalence classes under R on A partitions A . Conversely, if Π partitions A , then there exists an equivalence relation R on A s.t. each component of Π is the equivalence class of some $x \in A$ under R .

3. Show that equinumerosity is an equivalence relation on the class of all sets and thus partitions the class of all sets.
4. **Defn.** R partially orders A iff R is reflexive, transitive, and antisymmetric, meaning that for all $x, y \in A$, if $\langle x, y \rangle \in R$ and $\langle y, x \rangle \in R$, then $x = y$.

Define a relation that partially orders the collection of equivalence classes of \sim on the class of all sets.

5. Suppose A and B are both countable sets. Show that $A \cup B$ and $A \times B$ are both countable.
6. Prove that the set of all integers and the set of all rational numbers are both countable.
7. A member A of $\mathcal{P}(\mathbb{N})$ is said to be *cofinite* just in case $\mathbb{N} \setminus A$ is finite. Let B consist of the finite and cofinite members of $\mathcal{P}(\mathbb{N})$. Show that B is countable.
8. Prove that the open interval $(0, 1) \sim \mathbb{R}$.
9. In each case find an expression using only the non-logical parameter \in to express:
 - (a) $\forall x x \neq \{x\}$
 - (b) $\forall x x = \emptyset$.
10. Find an expression using only the non-logical parameter ' \in ' to express that no set has more than two elements. Write a general schema expressing that no set has more than n elements for $n > 1$.