Philosophy 422 | R. Rynasiewicz
Axiomatic Set Theory | Fall 2024

Exercise Set # 2

Due Thursday, September 19. (100 points, 10 each)

1. Extensionality together with Comprehension demands that the intersection of any two sets must exist. Do they also require that the union of any two sets must exist? If so, give a proof. If not, present a counterexample verifying that both Extensionality and Comprehension are satisfied and that there is a pair of sets whose union does not exist.

2. We gave a proof that Extensionality, Comprehension, Unordered Pairs, and Foundation together do not do not require any set to have more than two elements. In the proof, show by induction on n that both Extensionality and Comprehension are satisfied in each $\langle D_n, \epsilon^{\mathfrak{A}} \restriction D_n \rangle$. Why must they be satisfied in D, as well?

3. We also gave a proof that Extensionality, Comprehension, Unordered Pairs, and Foundation together do not rule out 3-cycles. In the proof, show by induction on n that Foundation is true for each $x \in G_n$. Why is Foundation true for all $x \in G$?

4. Show that

$$\text{Ext} + \text{UP} + \text{Compr} \vdash \forall x \forall y \forall u \forall v (\langle x, y \rangle = \langle u, v \rangle \leftrightarrow (x = u \ \wedge \ y = v)).$$

5. Find a model $\mathfrak{A}$ of Extensionality, Comprehension, and Unordered Pairs in which there exists $R \in |\mathfrak{A}|$ such that R is a relation but dom R does not exist. Do the same for ran R.

6. Show that $S(x) =_{df} x \cup \{x\}$ is well-defined.

7. Show that any sub-ordering of a well-ordering is a well-ordering, i.e., show that if $\langle A, R \rangle$ is a well-ordering and $B \subseteq A$, then $\langle B, R \rangle$ is a well-ordering.

8. Let $\langle A, R \rangle$ and $\langle B, S \rangle$ be well orderings. Prove in $\mathbf{ZF}^{-} - \mathbf{Inf} - \mathbf{P}$ that the lexicographic order on $A \times B$ well-orders $A \times B$.

9. Prove in $\mathbf{ZF}^{-} - \mathbf{Inf} - \mathbf{P}$ that if X is a set of ordinals, then
   (a) $\bigcup X$ is the least ordinal $\geq$ all elements of X, and
   (b) $\bigcap X$ is the least ordinal in X, assuming $X \neq 0$.

10. Show that the following two properties are independent.

    (a) x is transitive

    (b) $\in_x$ well-orders x