Philosophy 422 R. Rynasiewicz
Axiomatic Set Theory Fall 2024

Exercise Set # 3

Due Tuesday, October 15. [10 points each, 100 points total.]

1. *Successor and Limit Ordinals*. Derive the following in ZF^{-}−P.
   (a) $\forall\alpha\, ON(S(\alpha))$
   (b) $\forall\alpha\ \ \alpha < S(\alpha)$
   (c) $\forall\alpha\forall\beta(\beta < S(\alpha) \leftrightarrow \beta \leq \alpha)$
   (d) ω is the smallest limit ordinal.
2. *Basic properties of ordinal addition*. Establish the following in ZF^{-}−Inf−P.
   (a) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$
   (b) $\alpha + 0 = \alpha$
   (c) $\alpha + 1 = S(\alpha)$
   (d) $\alpha + S(\beta) = S(\alpha + \beta)$
   (e) If β is a limit ordinal, then $\alpha + \beta = \sup\{\alpha + \xi \mid \xi < \beta\}$.
3. *Failure of commutativity of ordinal addition.* Establish the following in ZF^{-}−P: if $\alpha \neq 0$, then $\omega < \omega + \alpha$, but if $\alpha < \omega$, then $\alpha + \omega = \omega$.
4. *Basic properties of ordinal multiplication*. Establish the following in ZF^{-}−Inf−P.
   (a) $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$
   (b) $\alpha \cdot 0 = 0$
   (c) $\alpha \cdot 1 = \alpha$
   (d) $\alpha \cdot S(\beta) = \alpha \cdot \beta + \alpha$
   (e) If β is a limit ordinal, then $\alpha \cdot \beta = \sup\{\alpha \cdot \xi \mid \xi < \beta\}$
   (f) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.
5. *Failure of commutativity of ordinal multiplication.* Suppose that α is an ordinal and $\alpha > 1$. Show that $\omega < \omega \cdot \alpha$ but that if $\alpha < \omega$, then $\alpha \cdot \omega = \omega$.
6. *Right distributivity for natural numbers.* Suppose that $\alpha, \beta, \gamma \in \mathbb{N}$. Establish that $(\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma$. *Hint.* Start by fixing α and β arbitrarily. Then do Peano induction on the parameter γ.

7. *Failure of right distributivity in the transfinite.* Suppose $0 < \alpha, \beta$. Prove that $\omega < \alpha \cdot \omega + \beta \cdot \omega$ but that if $\alpha, \beta < \omega$, then $(\alpha + \beta) \cdot \omega = \omega$.

8. Simplify the following.

   (a) $(\omega + 1) + \omega$

   (b) $\omega + \omega^2$

   (c) $(\omega + 1) \cdot \omega^2$

9. In ZF$^-$ - Inf - P show that $\alpha < \beta$ implies that $\gamma + \alpha < \gamma + \beta$ and $\alpha + \gamma \leq \beta + \gamma$. Give an example to show that '$\leq$' cannot be replaced by '$<$'. Also show that

$$\alpha \leq \beta \to \exists! \delta \ (\alpha + \delta = \beta).$$

10. Work in ZF$^-$ - Inf - P. Assume $\gamma \neq 0$. Show that if $\alpha < \beta$, then $\gamma \cdot \alpha < \gamma \cdot \beta$ and $\alpha \cdot \gamma \leq \beta \cdot \gamma$. Give an example to show that '$\leq$' cannot be replaced by '$<$'. Also, show that

$$(\alpha \leq \beta \wedge \alpha > 0) \to \exists! \delta, \xi (\xi < \alpha \ \wedge \ \beta = \alpha \cdot \delta + \xi).$$