Philosophy 422 | R. Rynasiewicz
Axiomatic Set Theory | Fall 2025

Exercise Set # 4

Due Tuesday, November 5.
[10 points each, 100 points total]

1. Show that if κ is a cardinal and α any ordinal, then
$$\alpha < \kappa \text{ iff } \alpha \prec \kappa,$$
but that it may fail that
$$\kappa < \alpha \text{ iff } \kappa \prec \alpha$$
unless α is also a cardinal.

2. Prove in **ZF**$^-$ the lemma $\forall\alpha\forall\beta(\alpha < \beta \to \omega_\alpha < \omega_\beta)$. (Hint: Take α to be arbitrary but fixed. Do transfinite induction on β.)

3. Show in **ZF**$^-$ that for any non-zero ordinal α,
   (a) ω_α is a successor cardinal iff α is a successor ordinal, and
   (b) ω_α is a limit cardinal iff α is a limit ordinal.

4. Establish in **ZF**$^-$+**AC** that for all cardinals κ, λ, and σ
   (a) $\kappa^{\lambda\oplus\sigma} = \kappa^\lambda \otimes \kappa^\sigma$, and
   (b) $\kappa^{\lambda\otimes\sigma} = \left(\kappa^\lambda\right)^\sigma$.

   (Hint. For (a) first show that if $B \cap C = \emptyset$, then ${}^{(B\cup C)}A \sim ({}^{B}A) \times ({}^{C}A)$. For (b) show that ${}^{C}\left({}^{B}A\right) \sim {}^{(C\times B)}A$ (a.k.a. Currying, after Haskel Curry).

5. Prove that ω is regular.

6. Let α be a limit ordinal. Show that the following are equivalent.
   (a) $\forall\beta, \gamma < \alpha\,(\beta + \gamma < \alpha)$.
   (b) $\forall\beta < \alpha\,(\beta + \alpha = \alpha)$.
   (c) $\forall X \subseteq \alpha\,(\mathit{type}(X) = \alpha \vee \mathit{type}(\alpha \setminus X) = \alpha)$.
   (d) $\exists\delta\,(\alpha = \omega^\delta)$ (ordinal exponentiation).

   Such α are called *indecomposable*.

7. Prove the Cantor normal form theorem for ordinals: Every non-0 ordinal α may be represented in the form:

$$\alpha = \omega^{\beta_1} \cdot \ell_1 + \cdots + \omega^{\beta_n} \cdot \ell_n,$$

where $1 \leq n < \omega$, $\alpha \geq \beta_1 > \cdots > \beta_n$, and $1 \leq \ell_i < \omega$. Furthermore, this representation is unique. α is called an *epsilon number* iff $n = 1$, $\ell_1 = 1$, and $\beta_1 = \alpha$ (i.e., $\omega^\alpha = \alpha$). Show that if κ is an uncountable cardinal, then κ is an epsilon number and there are κ epsilon numbers below κ; in particular, the first epsilon number, called ϵ_0 is countable. All exponentiation is ordinal exponentiation in this exercise.

8. (Kunen, problem 7, p. 43) Prove (in $\mathbf{ZF}^-$) that the following definition of ordinal exponentiation is equivalent to (Kunen's) Definition 9.5.
   Let

$$F(\alpha, \beta) = \{f \in {}^{\beta}\alpha : \ |\{\xi : f(\xi) \neq 0\}| < \omega\}.$$

   If $f, g \in F(\alpha, \beta)$ and $f \neq g$, say $f \lhd g$ iff $f(\xi) < g(\xi)$, where ξ is the largest ordinal s.t. $f(\xi) \neq g(\xi)$. Then $\alpha^\beta = \mathrm{type}(\langle F(\alpha, \beta), \lhd \rangle)$.

9. (Kunen problem 9, pp. 44-5.) Show in $\mathbf{ZF}^-$ that for any set X the following are equivalent.

   (a) X can be well-ordered.

   (b) There is a $C\colon (\mathcal{P}(X) \setminus \{0\}) \to X$ such that $\forall Y \subseteq \mathrm{X}(Y \neq 0 \to C(Y) \in Y)$.

   Hint: Fix some $p \notin X$ and let $C(Y) = p$ if $Y \notin (\mathcal{P}(X) \setminus \{0\})$. Define by transfinite recursion

$$F(\alpha) = C(X \setminus \{F(\xi) \mid \xi < \alpha\}).$$

10. (Kunen problem 10, p. 45.) Show in $\mathbf{ZF}^-$ that the following are equivalent.

    (a) Every set can be well-ordered.

    (b) $\forall \mathcal{S}(0 \notin \mathcal{S} \to \exists C(C\colon \mathcal{S} \to \bigcup \mathcal{S} \ \wedge \ (\forall Y \in \mathcal{S})(C(Y) \in Y)$.
    (This is the statement from which the Axiom of Choice gets its name.)