Philosophy 422 | R. Rynasiewicz
Axiomatic Set Theory | Fall 2024

Solutions to Exercise Set # 1

1. *Naive Set Theory*. Although Cantor founded transfinite set theory, he never stated axioms. Is there a reason we need axioms? What is Russell's paradox? How does ZF^- avoid it? (Kunen throughout uses ZF^- for ZF w/o the Axiom of Foundation.

   Soln. Nor did Cantor begin with a collection of definitions, as did Euclid. If a term or concept is to be defined, it has to be defined using other terms. To avoid circularity, it looks like one is stuck with an infinite regress.

   The terminology of mathematical theories must bottom out somewhere, and it bottoms out in an axiom system that *implicitly* defines a basis of terms from which the remaining can be explicitly defined. It turns out that an axiom system for a single relational predicate, *is an element of*, suffices.

   Unless axioms are stated, it is not clear which classes are sets. If one supposes that all things that satisfy a wff with one free variable (predicate) forms a set, then Russell's paradox arises. Consider the collection $C := \{x : x \notin x\}$. If C is a set, we can ask whether $C \in C$ or not. If $C \in C$, then the criterion for being an member of C fails to be satisfied, and so $C \notin C$. But if $C \notin C$, then the criterion *is* satisfied and so $C \in C$. Thus, $C \in C$ iff $C \notin C$.

   ZF avoids the paradox by permitting a predicate to pick out a set only as a subset of some already given set. (Comprehension Schema, a.k.a., Axiom of Separation).

   This may sound weak, since *prima facie* Russell's paradox can be run using classes rather than sets. But this is only if we understand that classes can be members of other classes. In ZF, this cannot be the case, since there a class is simply a predicate expression.

2. **Defn**. A relation R on a class A is said to be

   - *reflexive* iff for all $x \in A$, $\langle x, x\rangle \in R$,
   - *symmetric* iff for all $x, y \in A$, if $\langle x, y\rangle \in R$, then $\langle y, x\rangle \in R$,
   - *transitive* iff for all $x, y, z \in A$, if $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R$, then $\langle x, z\rangle \in R$,
   - an *equivalence relation* on A iff R is reflexive, symmetric, and transitive on A.

   Defn. Let R be an equivalence relation on A. Then

$$[x]_R =_{df} \{y \in A \mid \langle x, y\rangle \in R\}$$

is said to be the equivalence class of x under R on A.

Defn. A *partition* of a class A is a collection $\mathscr{C}$ of non-empty subclasses of A s.t. (i) the members of $\mathscr{C}$ are pairwise disjoint and (ii) $\bigcup \mathscr{C} = A$.

Establish the following.

Lemma. If R is an equivalence relation on A, then the collection $\{[x]_R \mid x \in A\}$ of equivalence classes under R on A partitions A. Conversely, if $\mathscr{C}$ partitions A, then there exists an equivalence relation R on A s.t. each component of $\mathscr{C}$ is the equivalence class of some $x \in A$ under R.

Proof. For the first part, (i) suppose that R is an equivalence relation on A. Let $[x]_R$ and $[y]_R$ be distinct equivalence classes. Suppose that $[x]_R \cap [y]_R \neq \emptyset$. Then there is a common member z, and hence $\langle x, z\rangle \in R$ and $\langle y, z\rangle \in R$. By symmetry, $\langle z, y\rangle \in R$, and hence, by transitivity, $\langle x, y\rangle \in R$. Thus $y \in [x]_R$, and so $[x]_R = [y]_R$, contrary to supposition. Therefore, $[x]_R \cap [y]_R = \emptyset$. (ii) Suppose $x \in A$. Then $x \in [x]_R$, and hence $x \in \bigcup\{[y]_R \mid y \in A\}$. Therefore, the set of equivalence classes induced by R on A partitions A.

For the second part, suppose that $\mathscr{C}$ partitions A. Define R such that for all $x, y \in A$: $\langle x, y\rangle \in R$ iff x and y are in the same component of $\mathscr{C}$. Trivially, R is reflexive. Almost as trivially, if $\langle x, y\rangle \in R$, then $\langle y, x\rangle \in R$. So R is symmetric. Finally, if $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R$, then $\langle x, z\rangle \in R$. So R is transitive. Furthermore, (i) if $[x]_R \neq [y]_R$, then $[x]_R \cap [y]_R = \emptyset$, and, of course, $\bigcup\{[x]_R \mid x \in A\} = A$. ∎

3. Show that equinumerosity is an equivalence relation on the class of all sets and thus partitions the class of all sets.

   Proof. Let A, B, C be sets. $A \sim A$ since the identity map on A is trivially bijective. Thus $\sim$ is reflexive. Suppose $A \sim B$. Then there is a bijection $f : A \to B$. Since the inverse of a bijection is bijective, $f^{-1} : B \to A$ is bijective. Thus $\sim$ is symmetric. Finally, suppose that $A \sim B$ and $B \sim C$. Then there are bijections $f : A \to B$ and $g : B \to C$. The composite map $g \circ f : A \to C$ is also bijective. For if $(g \circ f)(a) = (g \circ f)(a')$, i.e., $g(f(a) = g(f(a'))$, then $f(a) = f(a')$ given that g is injective, and in turn $a = a'$ given that f is injective. Thus, $g \circ f$ is injective. It is also surjective (onto). For, pick any $c \in C$. Since g is surjective, there exists $b \in B$ s.t. $g(b) = c$. Similarly, since f is surjective, there exists $a \in A$ such that $f(a) = b$. Thus, $(g \circ f)(a) = c$ and $g \circ f$ is onto. Hence $g \circ f$ is bijective, and so $A \sim C$. Therefore, $\sim$ is an equivalence relation on the class of all sets. ∎

4. **Defn**. R partially orders A iff R is reflexive, transitive, and antisymmetric, meaning that for all $x, y \in A$, if $\langle x, y\rangle \in R$ and $\langle y, x\rangle \in R$, then $x = y$.

Define a relation that partially orders the collection of equivalence classes of $\sim$ on the class of all sets.

Solution. Let R be the relation

$$R = \{\langle [x]_\sim, [y]_\sim \rangle \mid x \preceq y\}.$$

Since $x \preceq x$ for all sets x, $\langle [x]_\sim, [x]_\sim \rangle \in R$ and thus R is reflexive. For transitivity, suppose that $\langle [x]_\sim, [y]_\sim \rangle \in R$ and $\langle [y]_\sim, [z]_\sim \rangle \in R$. Then $x \preceq y$ and $y \preceq z$. Hence $x \preceq z$, and so $\langle [x]_\sim, [z]_\sim \rangle \in R$. Finally, for antisymmetry, suppose that $\langle [x]_\sim, [y]_\sim \rangle \in R$ and $\langle [y]_\sim, [x]_\sim \rangle \in R$. Then $x \preceq y$ and $y \preceq x$, and so, by the Schröder-Bernstein theorem, $x \sim y$. Thus, $[x]_\sim = [y]_\sim$ and R is antisymmetric. Therefore R partially orders the class of all equivalence classes of $\sim$. ■

5. Suppose A and B are both countable sets. Show that $A \cup B$ and $A \times B$ are both countable.

   Proof. Let $f : A \to \mathbb{N}$ and $g : B \to \mathbb{N}$ be injections. If their ranges do not overlap, then $f \cup g$ maps $A \cup B$ into $\mathbb{N}$. But what if the ranges overlap? Let $h : A \cup B \to \mathbb{N}$ be defined

$$h(x) = \begin{cases} 2 \cdot f(x) & \text{if } x \in A \\ 2 \cdot g(x) + 1 & \text{otherwise.} \end{cases}$$

   This is injective since both f and g are.

   For the Cartesian product, consider the map $h' : A \times B \to \mathbb{N} \times \mathbb{N}$ defined $h'(x, y) = \langle f(x), g(y) \rangle$. Again, h' is injective since both f and g are. Below (problem 4) we show that $\mathbb{N} \times \mathbb{N}$ is countable w/o assuming that, in general, if both A and B are countable, then so is $A \times B$. Thus, $A \times B$ is countable. ■

6. Prove that the set of all integers and the set of all rational numbers are both countable.

   Proof. For the case of the integers, define $f : \mathbb{Z} \to \mathbb{N}$ so that

$$f(p) = \begin{cases} 2p & \text{if } 0 \leq p \\ 2|p| - 1 & \text{otherwise.} \end{cases}$$

   Now, suppose that $f(p) = f(q)$. If $f(p)$ is even, then $0 \leq p, q$ and $2p = 2q$, and so $p = q$. If $f(p)$ is odd, then $p, q < 0$ and $2|p| - 1 = 2|q| - 1$. Thus, $|p| = |q|$, and, since p and q have the same sign, $p = q$. Therefore, f is injective and thus $\mathbb{Z}$ is countable.

   For the case of the rationals, clearly $\mathbb{Q} \preceq \mathbb{Z} \times \mathbb{Z}$ since $x \in \mathbb{Q}$ iff there exists $\langle p, q \rangle \in \mathbb{Z} \times \mathbb{Z}$ s.t. $x = p/q$. (So, define $f : \mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}$ by $f(x) = \langle p, q \rangle$ iff $x = p/q$ and p and q have no common divisor. Clearly, f is injective.) So, it suffices to show that $\mathbb{Z} \times \mathbb{Z} \preceq \mathbb{N}$. By the first part of this problem, it suffices in turn to show that $\mathbb{N} \times \mathbb{N} \preceq \mathbb{N}$. Let $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be given by $g(n, m) = 2^n 3^m$. By the fundamental theorem of arithmetic, this is injective (though hardly surjective). ■

If we want a bijection, we can inject $\mathbb{N}$ into $\mathbb{Z} \times \mathbb{Z}$ by the rule $n \mapsto (n, 0)$ and then with the two injections construct a bijection as indicated in the proof of the Schöder-Bernstein theorem. Or one might note that $\mathbb{Z} \times \mathbb{Z} \sim \mathbb{N} \times \mathbb{N}$ and then construct a bijection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by

$$f(n, m) = \begin{cases} n^2 + m & \text{if } m \leq n \\ m^2 + m + n & \text{otherwise.} \end{cases}$$

But it is not trivial to show that f is bijective.

7. A member A of $\mathscr{P}(\mathbb{N})$ is said to be *cofinite* just in case $\mathbb{N} \setminus A$ is finite. Let B consist of the finite and cofinite members of $\mathscr{P}(\mathbb{N})$. Show that B is countable.

   Proof. Start with the set $\mathscr{F}$ of all finite sets of natural numbers. We can easily show that that's countable by injecting it into $\mathbb{N}$. We do this as follows. Let $p_1, p_2, \ldots$ be an enumeration of the primes. For a given $A \in \mathscr{F}$ of cardinality m, Let $n_1, \ldots, n_m$ be the elements of A in increasing order. Now define $\phi : \mathscr{F} \to \mathbb{N}$ by

   $$\phi(A) = p_1^{\chi_A(n_1)+1} \cdots p_m^{\chi_A(n_m)+1}.$$

   (Recall that χ_A is the characteristic function of A.) By the fundamental theorem of arithmetic, ϕ is injective. Therefore $\mathscr{F} \preceq \mathbb{N}$. In fact, $\mathscr{F} \sim \mathbb{N}$ since the rule $n \mapsto \{n\}$ is injective, so $\mathbb{N} \preceq \mathscr{F}$ and we can apply the Schröder-Bernstein theorem.

   Let $\mathscr{C}$ be the set of cofinite sets. Clearly, $\mathscr{C} \sim \mathscr{F}$ by taking relative complements. So, we can biject them, respectively, onto the evens and the odds and then take the union. That maps B to $\mathbb{N}$ bijectively. ∎

8. Prove that $(0, 1) \sim \mathbb{R}$.

   Proof. We've already seen that $\mathscr{P}(\mathbb{N})$ is equinumerous with the open interval $(0, 1)$ of reals. So, it suffices to show that $(0, 1) \sim \mathbb{R}$. This can be done in two steps. Let $f : (0, 1) \to (-\pi/2, \pi/2)$ be the affine transformation $f(x) = \pi(x - 1/2)$, which is straightforwardly bijective. Next note that $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ is bijective. Composing this with f yields a map $h : (0, 1) \to \mathbb{R}$, explicitly given by $h(x) =_{df} \tan f(x)$. This is bijective since the composition of bijective functions is bijective. ∎

9. In each case find an expression using only the non-logical parameter $\in$ to express:

   (a) $\forall x \, x \neq \{x\}$

   (b) $\forall x \, x = \emptyset$

   Solns.

   (a) $\forall x (\exists! z (\forall y (y \in z \leftrightarrow y = x) \wedge x \neq z))$

(b) $\forall x(\exists! z(\forall y\, y \notin z) \wedge x = z)$

Many variants work as well.

10. Find an expression using only the non-logical parameter $\in$ to express that no set has more than two elements. Write a general schema expressing that no set has more than n elements for $n > 1$

Solutions. For the first, the following will do:

$$\forall x(\forall y_1 \forall y_2 \forall y_3((y_1 \in x \wedge y_2 \in x \wedge y_3 \in x) \rightarrow (y_1 = y_2 \vee y_1 = y_3 \vee y_2 = y_3)).$$

Also, the following:

$$\neg \exists x \exists y_1 \exists y_2 \exists y_3 (y_1 \in x \wedge y_2 \in x \wedge y_3 \in x \wedge y_1 \neq y_2 \wedge y_1 \neq y_3 \wedge y_2 \neq y_3).$$

For the second, generalizing on the first of the above:

$$\forall x \forall y_1 \cdots \forall y_{n+1} \left(\bigwedge_{1}^{n+1} y_i \in x \rightarrow \bigvee_{i<j} y_i = y_j \right).$$

And generalizing on the 2nd of the above:

$$\neg \exists x \exists y_1 \cdots \exists y_{n+1} \left(\bigwedge_{1}^{n+1} y_i \in x \rightarrow \bigvee_{i<j} y_i \neq y_j \right).$$