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Free oscillations of drops and bubbles: the initial-value problem

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We study the initial-value problem posed by the small-amplitude (linearized) free oscillations of free drops, gas bubbles, and drops in a host liquid when viscous effects cannot be neglected. It is found that the motion consists of modulated damped oscillations, with the damping parameter and frequency approaching only asymptotically the results of the normal-mode analysis. The connexion with the normal-mode method is demonstrated explicitly and the experimental relevance of our results is discussed.

1. Introduction

The small-amplitude oscillatory motion of drops and bubbles about the spherical shape is a classical problem in fluid mechanics considered in one form or another already by Kelvin (1890), Lamb 1932, pp. 473, 640 and Rayleigh (1894, art. 364). A number of other studies have also been devoted to this problem in recent times for its importance in chemical engineering (Valentine, Sather & Heideger 1965; Miller & Scriven 1968; Loshak & Byers 1973), spray cooling (Yao & Schrock 1976), multi-phase flow (Levich 1962; Delichatsios 1975), nuclear physics (Schoessow & Baumeister 1966; Wong 1976) and meteorology (Nelson & Gokhale 1972), as well as for its intrinsic scientific interest (Chandrasekhar 1959, 1961, p. 466; Reid 1960; Prosperetti 1979).

As with many other problems involving small-amplitude oscillatory flows, all existing theoretical investigations have been conducted by separation of the time variable, the so-called normal-mode technique (see, for example, Chandrasekhar 1961, p. 3). Although perfectly justified on mathematical grounds whenever the problem considered possesses a complete set of eigenvalues and eigenfunctions, this technique has very strong practical limitations in the solution of initial-value problems because of the complexity of the operations required. As a consequence, the transient regime of drop oscillations (and of many other systems as well) has never been considered, the existing results being limited essentially to the asymptotic values of the frequency and damping parameter. It is clear that, in the case of damped free oscillations, the value of this information is limited because the asymptotic regime may be reached so late that the motion has essentially died out.

In the present study we apply to the initial-value problem an alternative technique, based on the use of Laplace transforms, which has recently been developed (Prosperetti 1976; Menikoff *et al.* 1980; Prosperetti, Cucchiani & Dei Cas 1980; Cortelezzi & Prosperetti 1980). The connexion of this technique with the normal-mode result is explicitly established, and it is shown that the eigenvalue characteristic equation can be recovered from the complete transient result in a straightforward way.

The present mathematical approach is related to others used in the past for different types of initial-value problems. We may mention the decay of the motion of a floating body (Ursell 1964), the transient translational motion of a gas bubble (Chen 1974) and of solid bodies (Ockendon 1968) in a viscous liquid, and problems of interfacial stability (Bankoff 1961; Sekerka 1967).

In addition to its relevance for the specific system under consideration, this study will demonstrate some general features of the analysis and of the results obtainable for small-amplitude oscillatory flows with a free surface. In particular it is found that the motion consists of modulated oscillations, with varying frequency and damping parameter. The essential physical process responsible for this behaviour is the diffusion of vorticity from the boundaries into the bulk of the fluid(s), which leads to an integro-differential structure for the equation of motion of the amplitude of the oscillations.

The present study is based on results obtained in Prosperetti (1977) for a more general class of flows than those considered here. The reader is referred to that study for details of the solution of the fluid-mechanical problem. Here we shall confine our attention to the equation of motion for the oscillation amplitude. The two limiting cases in which one of the fluids has negligible dynamical effects (i.e. the free drop and the gas bubble) are considered separately. The last section contains some general remarks on the results and their relevance for the interpretation of experimental data.

2. The equation of motion of the free surface

The initial-value problem posed by the small-amplitude shape oscillations of a fluid droplet immersed in another unbounded fluid has been considered by Prosperetti (1977). In that work the interface separating the two fluids was assumed to have the form

$$r(\theta, \phi) = R + a_n(t)\, Y_n^m(\theta, \phi), \tag{1}$$

where R is the average equilibrium radius, Y_n^m is a spherical harmonic, and (r, θ, ϕ) are spherical co-ordinates centred at the centroid of the interface.† A solution of the linearized form of the Navier–Stokes equations was then sought, subject to the kinematic boundary condition stemming from (1), and to the dynamical boundary conditions on the tangential and the normal stresses at the interface. The special cases in which one of the fluids has negligible dynamical effects were considered in some detail. It was shown that in either case the amplitude $a_n(t)$ satisfies the following integro-differential equation

$$\ddot{a}_n + 2b_{n0}\dot{a}_n + \omega_{n0}^2 a_n + 2\beta_n b_{n0} \int_0^t Q_n(t-\tau)\, \dot{a}_n(\tau)\, d\tau = 0, \tag{2}$$

where dots denote differentiation with respect to time. For the case of a bubble (i.e. when the effects of the inner fluid are negligible) the function $Q_n(t)$ is defined by its Laplace transform $\tilde{Q}_n(p)$,

$$\tilde{Q}_n(p) = -[1 + \tfrac{1}{2}\tilde{\mathscr{K}}_{n-\frac{1}{2}}(q)]^{-1}, \tag{3}$$

† The superscript m is omitted from the amplitude $a_n(t)$ because it is found that, in the linearized approximation, the equation determining this quantity depends on the order, but not on the degree, of the spherical harmonic.

where $q = R(p/\nu)^{\frac{1}{2}}$ and $\tilde{\mathscr{K}}_{n-\frac{1}{2}}$ is a quotient of modified Bessel functions of the second kind,

$$\tilde{\mathscr{K}}_{n-\frac{1}{2}}(q) = qK_{n+\frac{1}{2}}(q)/K_{n-\frac{1}{2}}(q), \tag{4}$$

(see Onoe 1958). The other quantities appearing in (2) are specified as follows in terms of the liquid density ρ, kinematic viscosity $\nu = \mu/\rho$, and surface tension ζ,

$$b_{n0} = (n+2)(2n+1)\nu/R^2, \tag{5a}$$

$$\omega_{n0}^2 = (n-1)(n+1)(n+2)\zeta/\rho R^3, \tag{5b}$$

$$\beta_n = n(n+2)/(2n+1).$$

When it is the outer fluid to have negligible effects as would be the case for instance, for a free liquid droplet, one has in place of (3)–(5)

$$b_{n0} = (n-1)(2n+1)\nu/R^2, \quad \omega_{n0}^2 = n(n-1)(n+2)\zeta/\rho R^3, \tag{6a, b}$$

$$\beta_n = (n-1)(n+1)/(2n+1),$$

$$\tilde{Q}_n(p) = [1-\tfrac{1}{2}\mathscr{I}_{n+\frac{3}{2}}(q)]^{-1}, \tag{7}$$

where again $q = R(p/\nu)^{\frac{1}{2}}$ and

$$\mathscr{I}_{n+\frac{3}{2}}(q) = qI_{n+\frac{1}{2}}(q)/I_{n+\frac{3}{2}}(q) \tag{8}$$

where the I's are modified Bessel functions of the first kind.

Equation (2) is valid when the initial vorticity vanishes everywhere. In the more general case the right-hand side of this equation would acquire a forcing term dependent on the initial vorticity distribution.

It was shown in Prosperetti (1977) that the integral term in (2) is negligible as $t \to 0$, so that initially the motion is just that executed by a damped harmonic oscillator of natural frequency ω_{n0} and damping parameter b_{n0}. The expressions for these quantities given in (5) and (6) agree with the well-known results obtained from the irrotational approximation (see, for example, Lamb 1932, pp. 473, 639).

The more general case in which the dynamical effects of both fluids must be considered was also analysed in Prosperetti (1977), and although an equation corresponding to (2) was not explicitly given, it is quite readily derived following the same procedure and imposing the additional requirement of continuity of the tangential velocities. The result is

$$\ddot{a}_n + \Gamma_n^{-1}\int_0^t D_n(t-\tau)\dot{a}_n(\tau)\,d\tau + \omega_{n0}^2 a_n = 0, \tag{9}$$

where

$$\Gamma_n = n\rho_o + (n+1)\rho_i, \tag{10}$$

$$\omega_{n0}^2 = (n-1)n(n+1)(n+2)\zeta/\Gamma_n R^3, \tag{11}$$

$$\tilde{D}_n(p) = \frac{\left\{\begin{array}{l}[(2n+1)\mu_i\mathscr{I}_{n+\frac{3}{2}}(q_i) + 2n(n+2)(\mu_o-\mu_i)] \\ \quad\times[(2n+1)\mu_o\tilde{\mathscr{K}}_{n-\frac{1}{2}}(q_o) - 2(n-1)(n+1)(\mu_o-\mu_i)]\end{array}\right\}}{\mu_o\tilde{\mathscr{K}}_{n-\frac{1}{2}}(q_o) + \mu_i\mathscr{I}_{n+\frac{3}{2}}(q_i) + 2(\mu_o-\mu_i)}, \tag{12}$$

with $q_{i,o} = R(p/\nu_{i,o})^{\frac{1}{2}}$. In these equations quantities pertaining to the inner and outer fluids carry the subscripts i and o respectively. Notice that it is not possible to extract from the integral in (9) a constant which represents the initial damping as was done in (2) for the case of a single fluid. This feature derives from the much stronger source of vorticity at the free surface which is introduced by the no-slip condition applicable to the two-fluid case (see, for example, Prosperetti *et al.* 1980).

An analytical solution of (2) or (9) in the time domain appears to be beyond reach. The solution for the Laplace-transformed function $\tilde{a}_n(p)$ however is quite readily obtained as

$$\tilde{a}_n(p) = \frac{1}{p}\left[a_n(0) + \frac{\dot{a}_n(0)\,p - \omega_{n0}^2\,a_n(0)}{p^2 + 2b_{n0}\,p + \omega_{n0}^2 + 2\beta_n\,b_{n0}\,p\tilde{Q}_n(p)}\right], \tag{13}$$

for equation (2) and as

$$\tilde{a}_n(p) = \frac{1}{p}\left[a_n(0) + \frac{\dot{a}_n(0)\,p - \omega_{n0}^2\,a_n(0)}{p^2 + \Gamma_n^{-1}\,p\tilde{D}_n(p) + \omega_{n0}^2}\right], \tag{14}$$

for equation (9). The inversion of these transforms will be accomplished numerically. (Here $\dot{a}_n(0) = da_n(0)/dt$.)

The results of (13) and (14) refer to initial conditions for which the vorticity distribution in the liquid(s) vanishes. In the case of a non-zero initial vorticity the transform of the added non-homogeneous term in the right-hand side of (2) and (9) would appear in the numerator of the fractions in (13) and (14).

3. Asymptotic behaviour and the connexion with the normal-mode analysis

One may ask whether for large times equations (2) or (9) possess solutions behaving as $\exp(-\sigma_\infty t)$, with σ_∞ a complex constant. The answer may be obtained quite readily from a consideration of the transformed solutions (13) and (14) because, if $a_n(t) = \exp(-\sigma_\infty t)\,v_n(t)$ with $v_n(t) \to$ constant for $t \to \infty$, known properties of the Laplace transform imply that

$$\lim_{p\to 0} p\tilde{v}_n(p) = \lim_{p\to 0} p\tilde{a}_n(p - \sigma_\infty) = \text{constant}, \tag{15}$$

(Widder 1941, cha. 5; see also Prosperetti 1976) which is only possible if the denominators in (13) and (14) vanish when $(-\sigma_\infty)$ is substituted for p. For the case of a free droplet we obtain in this way

$$\sigma_\infty^2 - 2b_{n0}\,\sigma_\infty + \omega_{n0}^2 - 2\beta_n\,b_{n0}\,\sigma_\infty\,[1 - \tfrac{1}{2}\mathscr{J}_{n+\frac{3}{2}}(x)]^{-1} = 0, \tag{16a}$$

where $x = R(\sigma_\infty/\nu)^{\frac{1}{2}}$ and $\mathscr{J}_{n+\frac{3}{2}}(x) = xJ_{n+\frac{1}{2}}(x)/J_{n+\frac{3}{2}}(x)$. Equation (16*a*) coincides with the characteristic equation obtained by Chandrasekhar (1959, 1961, p. 466) and Reid (1960) by means of the normal-mode analysis.

The same argument applied to the bubble case gives the following equation

$$\sigma_\infty^2 - 2b_{n0}\,\sigma_\infty + \omega_{n0}^2 + 2\beta_n\,b_{n0}\,\sigma_\infty[1 + \tfrac{1}{2}\mathscr{H}_{n-\frac{1}{2}}(x)]^{-1} = 0, \tag{16b}$$

where $\mathscr{H}_{n-\frac{1}{2}}(x) = xH_{n+\frac{1}{2}}(x)/H_{n-\frac{1}{2}}(x)$, and the H's are Hankel functions of the first or second kind.† On the basis of the known properties of these functions (see, for example, Erdélyi *et al.* 1953, p. 78) it is straightforward to conclude that (16*b*) may be written as the quotient of two polynomials, so that there is only a finite number of σ_∞'s obtainable in this way. However, it is easy to prove that a continuous spectrum exists consisting of the entire positive real semi-axis of the σ plane (Prosperetti 1980).

† The first kind functions give rise to the normal modes such that $\text{Im}\,\sigma_\infty > 0$, the second kind ones to those with $\text{Im}\,\sigma_\infty < 0$ (see Prosperetti 1980).

Mathematically, the existence of this continuous spectrum is associated with the fact that for the bubble case the complex plane on which the function (13) is defined must be cut along the real negative semi-axis to avoid ambiguities in the definition of the function $\mathscr{H}_{n-\frac{1}{2}}(q)$. This procedure is necessary also in the two-liquid case, but not in the free drop one because $\mathscr{J}_{n+\frac{3}{2}}(-q) = \mathscr{J}_{n+\frac{3}{2}}(q)$. Our result for the discrete spectrum, (16*b*), may be proved to coincide with the characteristic equation given, in a somewhat more complicated form, by Miller & Scriven (1968). These authors however disregarded entirely the continuous spectrum.

Similar results are found for the two fluid case. From equations (12) and (14) we derive the following characteristic equation

$$\begin{aligned}(\sigma_\infty^2+\omega_{n0}^2)\,\Gamma_n[\mu_\mathrm{o}\mathscr{H}_{n-\frac{1}{2}}(x_\mathrm{o})+\mu_\mathrm{i}\mathscr{J}_{n+\frac{3}{2}}(x_\mathrm{i})+2(\mu_\mathrm{o}-\mu_\mathrm{i})]\\ -\sigma_\infty[(2n+1)\mu_\mathrm{i}\mathscr{J}_{n+\frac{3}{2}}(x_\mathrm{i})+2n(n+2)(\mu_\mathrm{o}-\mu_\mathrm{i})]\\ \times[(2n+1)\mu_\mathrm{o}\mathscr{H}_{n-\frac{1}{2}}(x_\mathrm{o})-2(n-1)(n+1)(\mu_\mathrm{o}-\mu_\mathrm{i})]=0,\end{aligned}\tag{17}$$

where $x_{\mathrm{i,\,o}} = R(\sigma_\infty/\nu_{\mathrm{i,\,o}})^{\frac{1}{2}}$. Equations (16*a*, *b*) are contained as limiting cases in this result.

The connexion with the normal-mode approach can be made more rigorous and formally more satisfactory with the aid of the inversion theorem for the Laplace transform and of the calculus of residues. In this way one would find an expression of the form

$$a_n(t) = \sum_k c_k \exp(-\sigma_{\infty,\,k}t) + \int_{-\infty}^{\lambda} F(x)\,e^{xt}\,dx,$$

which has the structure of the summation over the discrete and the continuous spectrum to which the normal-mode analysis would in principle lead. From this point of view equation (15) is just the condition to ensure that $\sigma_{\infty,\,k}$ be a pole for $\tilde{a}_n(p)$, and the constants c_k are found to be given by

$$c_k = \frac{\dot{a}_n(0)\,\sigma_{\infty,\,k}+\omega_{n0}^2\,a_n(0)}{\sigma_{\infty,\,k}[d\Delta_n/dp]_{p=-\sigma_{\infty,k}}},\tag{18}$$

where $\Delta_n(p)$ is the denominator in (13) or (14). (In a normal-mode framework the constants c_k should be computed by means of scalar products in a suitable Hilbert space.) For cases of non-zero initial vorticity only the numerator of the fraction in (18) would be altered.

In the next section we shall consider the two one-liquid cases. The general case will be resumed in the following section.

4. Shape oscillations of free drops and of gas bubbles

We may summarize the preceding results for the two one-liquid cases (i.e. the bubble and the free drop) by saying that the system behaves as a damped oscillator characterized initially by the natural frequency $(\omega_{n0}^2-b_{n0}^2)^{\frac{1}{2}}$ and the damping parameter b_{n0}, and asymptotically by the natural frequency $\mathrm{Im}\,\sigma_\infty$ and the damping parameter $\mathrm{Re}\,\sigma_\infty$. Here σ_∞ is one of the normal modes given by (16), which in general will be the one with the smallest real part: we shall refer to this mode as to the first normal mode and from now on the symbol σ_∞ will be used to denote it. It is of interest

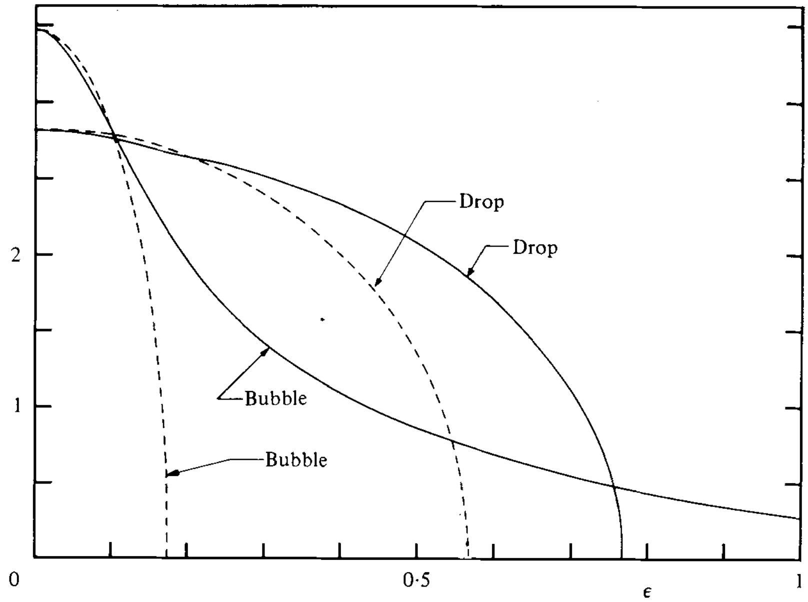

FIGURE 1. Dimensionless asymptotic (——, equations (16a, b)) and initial irrotational (---, equations (5b) and (6b)) oscillation frequency as a function of the dimensionless viscosity for free drops and gas bubbles for $n = 2$.

to compare these quantities, because the greater their difference, the more the dynamical behaviour of the system differs from that of a simple oscillator. For convenience we introduce the following non-dimensional quantities

$$B_{n0} = (R^3\rho/\zeta)^{\frac{1}{2}} b_{n0}, \quad \Omega_{n0} = (R^3\rho/\zeta)^{\frac{1}{2}}(\omega_{n0}^2 - b_{n0}^2)^{\frac{1}{2}}, \tag{19}$$

$$B_{n\infty} = (R^3\rho/\zeta)^{\frac{1}{2}} \operatorname{Re}\sigma_\infty, \quad \Omega_{n\infty} = (R^3\rho/\zeta)^{\frac{1}{2}} \operatorname{Im}\sigma_\infty. \tag{20}$$

Equations (5a) and (6a) give then respectively

$$B_{n0} = (n+2)(2n+1)\epsilon \quad \text{(bubble)}, \tag{21a}$$

$$B_{n0} = (n-1)(2n+1)\epsilon \quad \text{(drop)}, \tag{21b}$$

where the parameter ϵ, which may be considered a dimensionless viscosity, is given by

$$\epsilon = \nu(\rho/R\zeta)^{\frac{1}{2}}. \tag{22}$$

The proof that $(B_{n\infty}, \Omega_{n\infty}) \to (B_{n0}, \Omega_{n0})$ as $\epsilon \to 0$ has been given by Chandrasekhar (1959, 1961) and Reid (1960) for the drop case and by Miller & Scriven (1968) for the bubble case, and will not be repeated here. We give graphs of the quantities defined in (19) and (20) as functions of the parameter ϵ in figures 1 and 2 for $n = 2$; further results of this type will be found in Prosperetti (1980).

Figure 1 shows the behaviour of Ω_{20} and of $\Omega_{2\infty}$. It is observed that the initial and the asymptotic frequencies in the drop case vanish for $\epsilon = 2\sqrt{2}/5 \simeq 0{\cdot}5657$ and $\epsilon \simeq 0{\cdot}7665$ respectively. This behaviour corresponds to the transition from periodic to aperiodic decay of the oscillations. It is interesting to notice that this transition

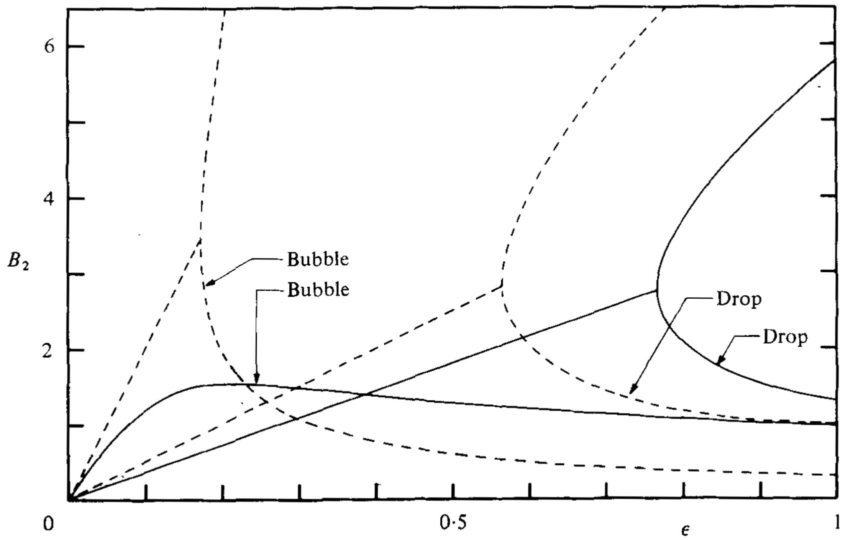

FIGURE 2. Dimensionless asymptotic (——, equations (16*a*, *b*)) and initial irrotational (– – –, equations (5*a*) and (6*a*)) damping parameter as a function of the dimensionless viscosity for free drops and gas bubbles for $n = 2$.

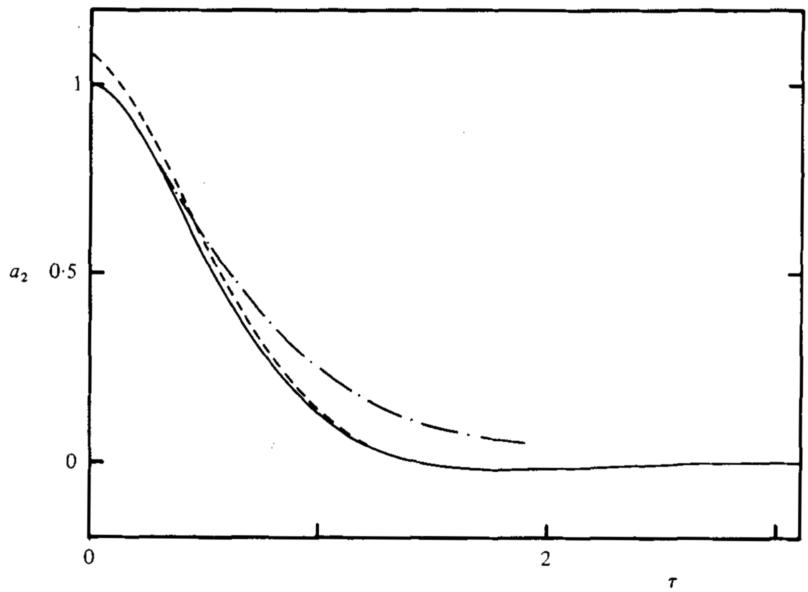

FIGURE 3. Oscillation amplitude of a free drop for $n = 2$, $\epsilon = 0{\cdot}6$, $a_2(0) = 1$, $\dot{a}_2(0) = 0$, obtained from the numerical inversion of the result (13) (——) compared with the least-damped normal mode (– – –) and the initial, irrotational approximation (–·–·–).

occurs much earlier for Ω_{20} than for $\Omega_{2\infty}$. Thus in a certain range of values of ϵ it is possible for the motion to start out as an aperiodic relaxation and to evolve into periodic oscillations as vorticity smooths out the large velocity gradients associated with the initial irrotational flow. An example of this behaviour is shown in figure 3. For the bubble case the transition to aperiodic motion occurs for $\epsilon = \sqrt{3}/10 \simeq 0{\cdot}1732$ for Ω_{20},

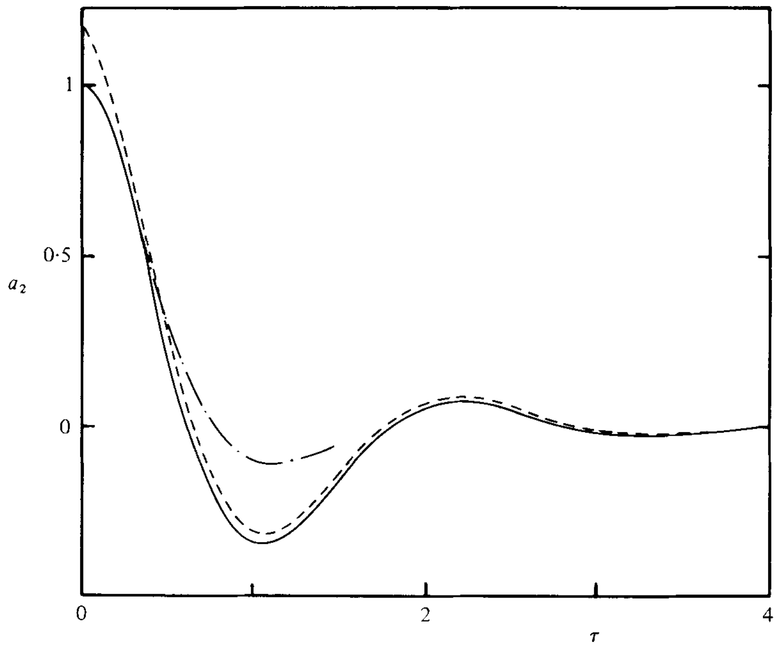

FIGURE 4. Oscillation amplitude of a free drop for $n = 2$, $\epsilon = 0{\cdot}1$, $a_2(0) = 1$, $\dot{a}_2(0) = 0$ obtained from the numerical inversion of the complete result (13) (——) compared with the least-damped normal mode (----) and the initial, irrotational approximation (-·--·-).

but only for $\epsilon \to \infty$ for $\Omega_{2\infty}$. Thus, no matter how large the viscosity, the bubble motion will always have an oscillatory 'tail', except in the limit $\epsilon \to \infty$. The physical reason for this difference between the drop and bubble cases is that in an unbounded domain the velocity gradients can become arbitrarily small.

Figure 2 shows B_{20} and $B_{2\infty}$ as functions of ϵ. The curves for the drop case exhibit a bifurcation point for those values of ϵ at which Ω_{20} and $\Omega_{2\infty}$ disappear. Beyond this point the normal modes become purely real, the lower one corresponding to the 'creeping motion' of a strongly overdamped oscillator (Chandrasekhar 1961). In the bubble case the transition to non-oscillatory behaviour is exhibited only by B_{20}. In place of the sharp maximum that the other curves exhibit at the bifurcation point, $B_{2\infty}$ exhibits a much milder maximum after which it starts slowly to decrease. As could be anticipated, in all cases the maxima occur approximately for values of ϵ such that the diffusion length in the course of one oscillation, which is of the order of $(\nu/\omega_{20})^{\frac{1}{2}}$, equals the characteristic length R.

Figures 3 to 5 show the results obtained by the numerical inversion of the Laplace transforms (13) for the drop and bubble cases (continuous lines). The numerical method of Durbin (1974) has been used, which has been found sufficiently stable for the time intervals considered. In all the examples presented in this and in the following section the comparison with the normal mode results (dashed lines) has been made choosing as initial condition for the latter the value given by (18). Clearly this particular choice has the effect of making the difference with the complete solution vanishingly small for $t \to \infty$, but introduces a discrepancy in the initial stages. Conversely, if agreement with the exact solution were imposed for $t \to 0$, a difference would appear

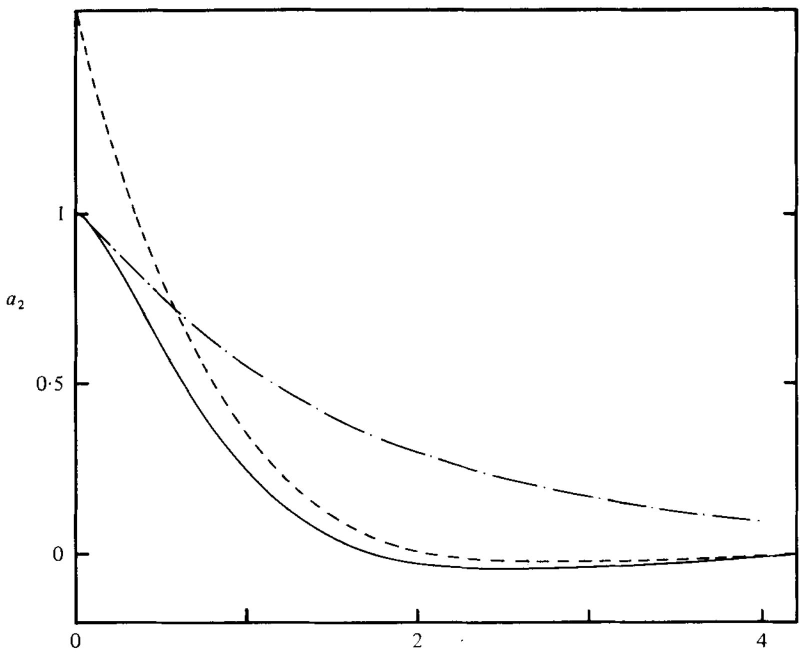

FIGURE 5. Oscillation amplitude of a gas bubble for $n = 2$, $\epsilon = 0{\cdot}5$, $a_2(0) = 1$, $\dot{a}_2(0) = 0$ obtained from the numerical inversion of the complete result (13) (——) compared with the least-damped normal mode (----) and the initial, irrotational approximation (–·–·–).

for large times (see Prosperetti 1976 for some examples). In figures 3 to 5 we also show using dash-and-dot-lines the results obtained by means of the irrotational approximation (Lamb 1932, p. 640). As is clear from the preceding discussion this approximation is uniformly valid in time for small ϵ, but is valid initially for any ϵ. The figures are in terms of a dimensionless time τ defined by

$$\tau = (\zeta/\rho R^3)^{\frac{1}{2}} t,$$

and the initial conditions $a_n(0) = 1$, $\dot{a}_n(0) = 0$ have been used.

Figure 3 refers to the drop case for $n = 2$ and $\epsilon = 0{\cdot}6$. This value of ϵ is greater than the critical one as far as the initial motion is concerned, but smaller than the critical one for $(B_{2\infty}, \Omega_{2\infty})$. The appearance of oscillations for sufficiently large times can be discerned in the figure. Figures 4 and 5 are for the bubble case for $\epsilon = 0{\cdot}1$ and $\epsilon = 0{\cdot}5$ respectively. The first value is smaller than the critical one, while the other one is much larger. It is seen that the differences between the three sets of curves are much more pronounced in the bubble than in the drop cases. Again, this feature is a consequence of the unboundedness of the domain in the former case.

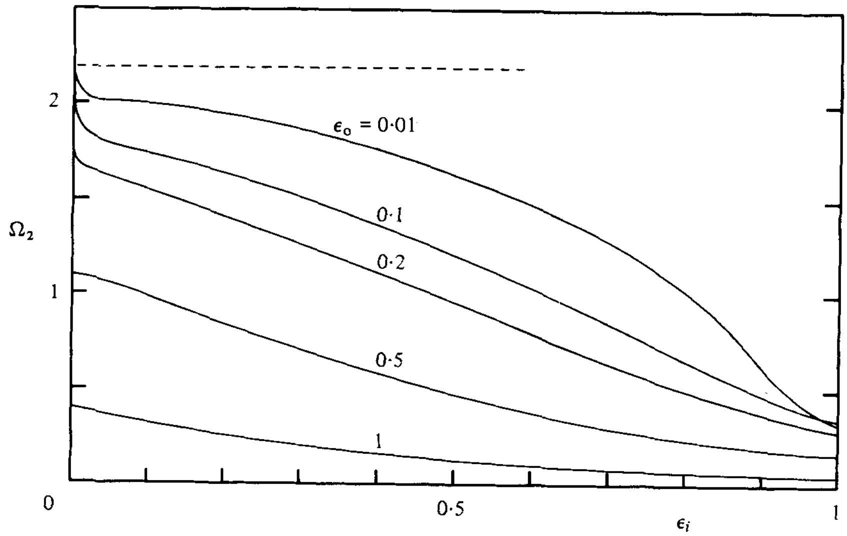

FIGURE 6. Dimensionless asymptotic oscillation frequency of a liquid droplet of dimensionless viscosity ϵ_i in a liquid of dimensionless viscosity ϵ_o for $n = 2$ (from (17)). Both liquids have the same density. The dashed line is the irrotational inviscid result (11).

5. Liquid drop in a viscous liquid

We now turn to the more general case in which both fluids have non-negligible dynamical effects as would happen for a liquid droplet immersed in another immiscible liquid. In view of the fact that our results apply only to the case of no relative motion between the droplet and the host liquid (which is known to influence the process, see Subramanyam 1969) we shall consider here only the neutrally buoyant case $\rho_i = \rho_o$. The case of unequal densities could be of greater interest in a micro-gravity environment such as that provided by Spacelab, but the general features of the motion would be very similar to those illustrated here. Normal-mode results for unequal densities can be found in Prosperetti (1980).

Figures 6 and 7 are plots of $\Omega_{2\infty}$ and $B_{2\infty}$, as determined from (17), as functions of

$$\epsilon_i = \nu_i(\rho_i/R\zeta)^{\frac{1}{2}}, \quad \epsilon_o = \nu_o(\rho_o/R\zeta)^{\frac{1}{2}}.$$

The non-dimensionalization of B and Ω has been made as in (20) on the basis of the physical properties of the inner fluid. The dashed line in figure 6 corresponds to the inviscid result (11). It is clear that this value is a very poor approximation except for extremely small values of ϵ_i and ϵ_o. The strong effect of the continuity of the tangential velocity is very clearly illustrated by the sharp dependence on ϵ_i for small values of this quantity that the curves for Ω and B exhibit in correspondence of $\epsilon_0 = 0{\cdot}01$, $0{\cdot}1$, and $0{\cdot}2$.

Figures 8 and 9 show two examples of the amplitude a_2 as a function of the dimensionless time $\tau = (\zeta/\rho_i R^3)^{\frac{1}{2}}t$ for $\epsilon_i = 0{\cdot}1$, $\epsilon_o = 0{\cdot}5$ and for $\epsilon_i = 0{\cdot}5$, $\epsilon_o = 0{\cdot}1$ respectively. The dashed lines are the asymptotic solutions with the initial condition chosen as before so as to ensure agreement for $t \to \infty$. Although no explicit form of an initially valid solution in the sense of the previous section can be given here, it is clear from the

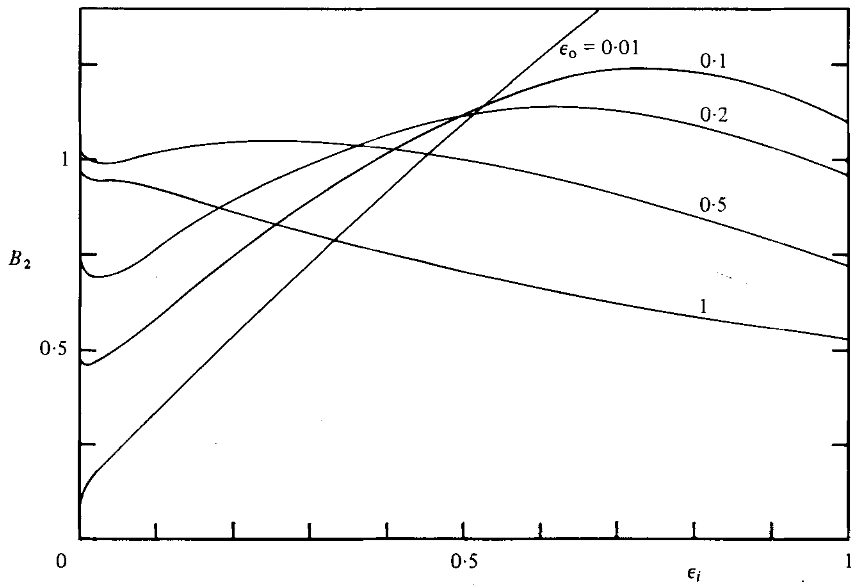

FIGURE 7. Dimensionless asymptotic damping parameter of a liquid droplet of dimensionless viscosity ϵ_i in a liquid of dimensionless viscosity ϵ_o for $n = 2$ (from (17)). Both liquids have the same density.

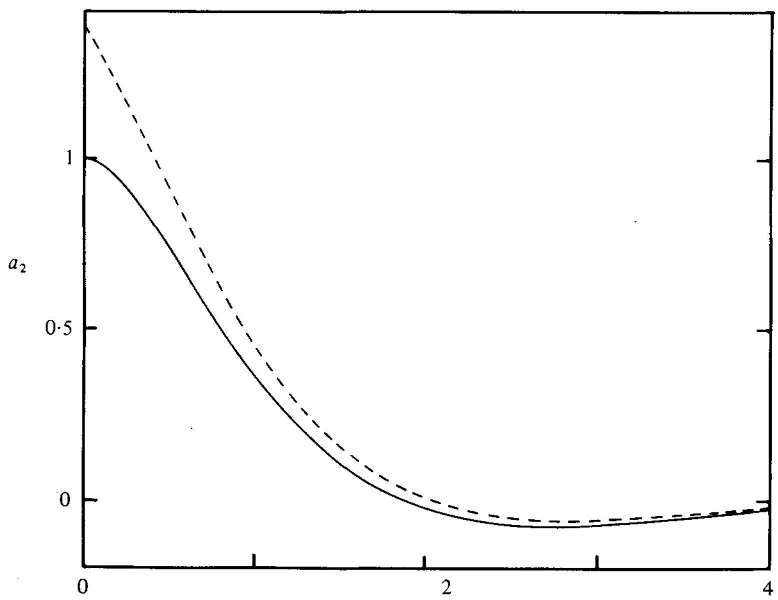

FIGURE 8. Oscillation amplitude of a viscous liquid drop in a viscous liquid of equal density for $n = 2$, $\epsilon_i = 0{\cdot}1$, $\epsilon_o = 0{\cdot}5$, $a_2(0) = 1$, $\dot{a}_2(0) = 0$ obtained from the numerical inversion of the complete result (14) (——) compared with the least-damped normal mode (– – – –).

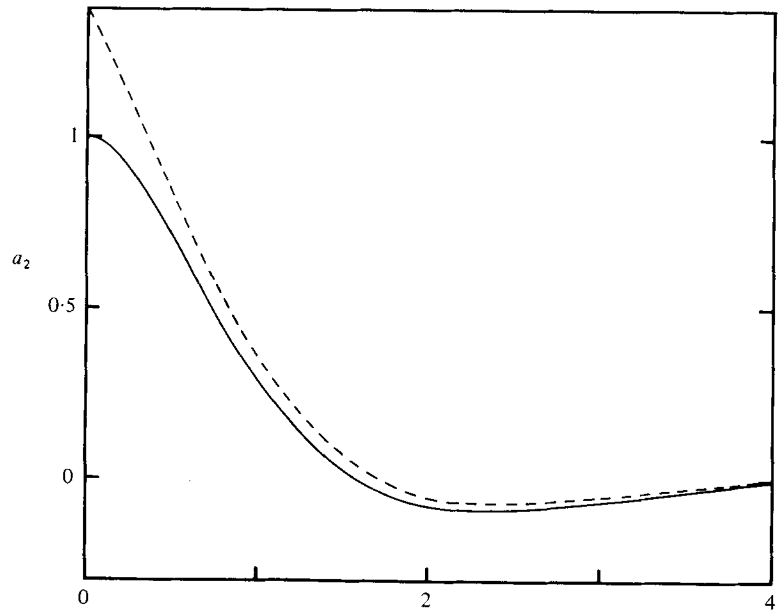

FIGURE 9. Oscillation amplitude of a viscous liquid drop in a viscous liquid of equal density for $n = 2$, $\epsilon_i = 0{\cdot}5$, $\epsilon_o = 0{\cdot}1$, $a_2(0) = 1$, $\dot{a}_2(0) = 0$ obtained from the numerical inversion of the complete result (14) (——) compared with the least-damped normal mode (- - - -).

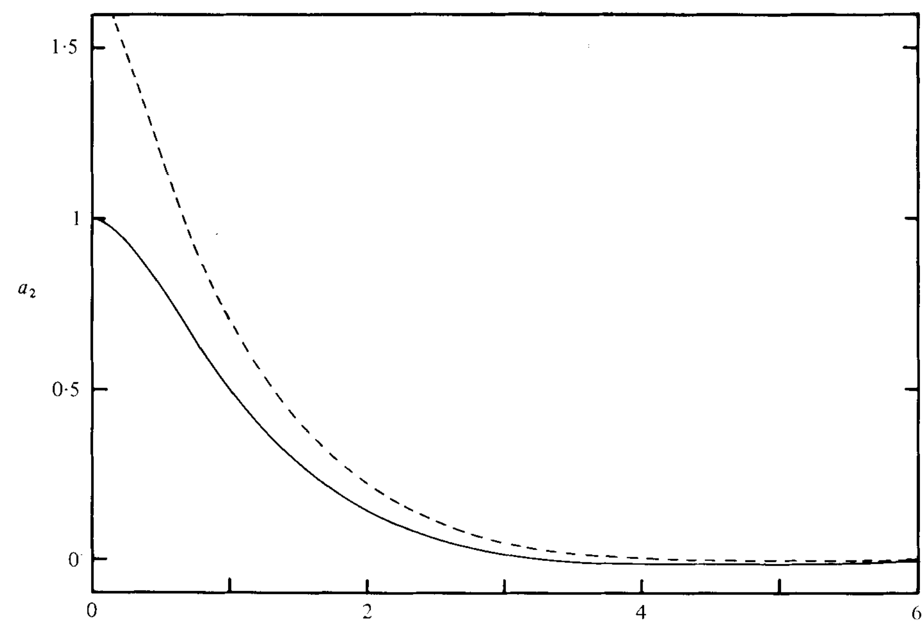

FIGURE 10. Oscillation amplitude of a viscous liquid drop in a viscous liquid of equal density for $n = 2$, $\epsilon_i = \epsilon_o = 0{\cdot}5$, $a_2(0) = 1$, $\dot{a}_2(0) = 0$ obtained from the numerical inversion of the complete result (14) (——) compared with the least-damped normal mode (- - - -).

fact that $da/d\tau$ does not vanish in $\tau = 0$ for the asymptotic solution that again we have a difference between the 'initial' and asymptotic frequency. A similar remark applies also to our last example, shown in figure 10, relative to a strongly damped case, $\epsilon_i = \epsilon_0 = 0{\cdot}5$.

6. Conclusions

In this study we have shown that the free oscillations of droplets and bubbles about the spherical shape cannot be represented in terms of a single value of the frequency and of the damping parameter. Rather, a representation in terms of modulated oscillations of the form

$$a_n(t) \propto \exp\left[-b(t) \pm i\omega(t)\right] t, \tag{23}$$

would appear to represent more closely the process, both mathematically and physically. The asymptotic values of ω and b for $t \to \infty$ are those given by the normal-mode analysis. Explicit expressions of these quantities for $t \to 0$ are available only for the two cases in which only one fluid has significant dynamical effects (i.e. the free drop and the gas bubble). The question of determining explicitly the time dependence of b and ω is a very interesting one, which however must be left open at this time in view of the complicated structure of the exact results (13) and (14). Some modified version of the two-timing technique might perhaps lead to approximate expressions when the parameter ϵ is small.

From a practical viewpoint the time dependence of b and ω has the obvious consequence that if an attempt is made to interpret experimental data in terms of a single value for these quantities, differences would be observed between one oscillation and the following one. Procedures such as averaging of these data would then lead to incorrect results. This remark is important in view of the fact that observation of the free oscillations of drops and bubbles has been proposed as a means to infer the rheological properties of fluid-fluid interfaces (Miller & Scriven 1968; Ramabhadran, Byers & Friedly 1976). In view of our results it would appear that measurements on forced oscillations such as those reported by Marston & Apfel (1979) would be more suitable because in that case the response of the system contains essentially only the least damped normal mode.

In conclusion a comment is in order on the validity of the linearized approximation on the basis of which the above results have been derived. A straightforward estimate of the order of magnitude of the convective term in the momentum equation shows that it will be negligible compared with $\partial u/\partial t$ provided that $|a_n(t)| \ll R$, as was to be expected. When this condition is satisfied second-order effects, such as microstreaming (see, for example, Riley 1967) are negligible. More interesting is perhaps the question of how the 'memory' effect exhibited by the motion under consideration would be altered by convective momentum transport. Since surface particles are forced to remain on the interface by the kinematic boundary condition, the transport of vorticity into the body of the fluids is also in the non-linear case strongly influenced by diffusion, which is the process responsible for the integro-differential structure of equations (2) and (9). However, the inclusion of non-linear terms would certainly change the vorticity source at the interface and its distribution in the fluids, so as to affect the 'modulation' of the quantities $b(t)$ and $\omega(t)$ appearing in (23).

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